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ARTICLE

Effective Bandwidth Estimation in Data Networks: An Analysis for Two Traffic Characterizations

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ARTICLE INFO

Article history
Received: 15 May 2021
Accepted: 23 June 2021
Published Online: 28 June 2021

Keywords:
Effective bandwidth
Markov fluid model
Kernel estimation
Data networking
Monte Carlo Markov Chain algorithms

ABSTRACT

The Generalized Markov Fluid Model (GMFM) is assumed for modeling sources in the network because it is versatile to describe the traffic fluctuations. In order to estimate resources allocations or in other words the channel occupation of each source, the concept of effective bandwidth (EB) proposed by Kelly is used. In this paper we use an expression to determine the EB for this model which is of particular interest because it allows expressing said magnitude depending on the parameters of the model. This paper provides EB estimates for this model applying Kernel Estimation techniques in data networking. In particular we will study two differentiated cases: dispatches following a Gaussian and Exponential distribution. The performance of the proposed method is analyzed using simulated traffic traces generated by Monte Carlo Markov Chain algorithms. The estimation process worked much better in the Gaussian distribution case than in the Exponential one.

1. Introduction

The need to aggregate several services in a telecommunication network leads to the emergence of the concept of integrated services digital network. Integration means that the network is able to transport many kinds of information as voice, video, data, all of them in digital form, using a single infrastructure. Therefore, the technical problem that motivates this work is the question of working in an environment of shared resources.

Variable Bit Rate (VBR) font multiplexing poses a mathematical and statistical problem: estimating the resource requirements of a font or set of fonts and, as sources are variable, statistical gain is to be expected. Through statistical multiplexing, the different requirements of each service can be explored during the connection. The development of statistical tools for studying the behavior of a network link arises with greater force from the notion of effective bandwidth introduced by Kelly in 1996 [5], which allows finding expressions to estimate the probability of loss in a link. The EB concept can be applied to sources or to aggregated traffic, as it can be the networks core link, but also it can be used for any shared resource models.

The price to pay for multiplexing sources is that the probability that many sources decide to dispatch the maximum rate, in which case there would be an overflow, is not zero. To minimize the effects of data loss and maintain quality of service (QoS) for both, current and future sources, it is necessary to have mechanisms of admission control that can decide whether to accept a new connection.

For this we need mathematical models to describe the behavior of the sources. Making models traffic carried by

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 DOI: https://doi.org/10.30564/ese.v3i1.3368

the network services is a necessary goal for dimensioning of its components and to evaluate its performance. Through traffic models, the appropriate descriptors can be found that characterize the service and facilitate management tasks such as establishing admission control criteria (CAC).

This paper is structured as follows. Section 2 introduces the Generalized Markov Fluid Model, and provides an expression to determine the EB for this model, a tool that will be used to measure the channel occupancy of each source. Section 3 studies the kernel estimation technique, its scope and properties, with the objective of using this tool to estimate the EB for our model. Section 4 presents the parameters of the simulated model and provides kernel estimates of the EB of the GMFM from traces, for two different cases: dispatches following a Gaussian and Exponential distribution. Conclusions are drawn in Section 5, together with some considerations on future work.

2. Mathematical Model and Calculation of the EB

2.1 Model

Within the most user-friendly source models, Markov models are used as they capture temporal correlation. These processes are characterized by a set of states, which form a Markov chain, and the transition times between them. In our case we use the Generalized Markov Fluid Model, introduced in [1].

This model is modulated by a continuous time, homogeneous and irreducible Markov chain, and in each state of the chain, the generation rate is a random variable, distributed according to a probability law f_i, that do not change during the time interval in which the Markov chain is in that state.

To interpret the model, we could think that each state in the chain is interpreted as the activity performed by a user, like chat or video conferences, so an abrupt change in the transfer speed report a change of state in the chain. Within a state, the speed data transfer assumes values that depend specifically for such activity, according to some probability distribution.

2.2 Effective Bandwidth

When variable rate sources are multiplexed on a link, a capacity greater than the average rate but less than the maximum transmission rate is reserved for each one. Indeed, the mean rate would be a too optimistic estimation, that would cause frequent losses, and on the other side, the peak rate would be too pessimistic and would lead to a resource waste. Effective bandwidth defined by F. Kelly in [5] is a measure, useful and realistic, of channel occupancy.

In order to estimate EB for a given GMFM, formulas of the type obtained by Kesidis, Walrand and Chang [6] were obtained. The advantage of this type of formula is that its parameters can be estimated from traffic traces.

Let us consider $\{X_t\}_{t\geq 0}$ a GMFM, then the effective bandwidth has the following expression:

$$\alpha(s,t) = \frac{1}{st}\log\{\pi exp[(Q+sH)t]1\} \tag{1}$$

where 1 is a column vector with all entries equal to 1, π is the invariant distribution, Q the infinitesimal generator for the modulating chain and H is a diagonal matrix of dimension k, whose non-zero elements are the first moments, μ_i, of the law governing the generation rate in state i.

The importance of this result is that provides an expression for the EB that depends on elements that can be estimated with traffic traces, like the infinitesimal generator of the modulating chain, its invariant distribution and the average transfer rate. The properties of this estimator can be seen in [1].

3. Invariant Distribution Estimation

In this section the estimators $\widehat{Q}, \widehat{H}$ and $\widehat{\pi}$ of the parameters in (1) are calculated using kernel methods techniques.

3.1 Kernel Density Estimation Methods

Given a simple random sample $X_1, \dots, X_n$ of the random variable of interest with density, the most common method in nonparametric density estimation is the so-called kernel-type estimator, used since the 1960s and whose expression is as follows:

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right) \tag{2}$$

where $h > 0$ is the window (or smoothing parameter), and K is a kernel function, i.e., a non-negative bounded and real-valued integrable function with $\int K = 1$, unimodal and symmetric around .

The estimator (2) has two unknown elements. On the one hand, the parameter usually referred to as bandwidth, but which in this paper we will call it window size to avoid confusion, whose choice will significantly affect the estimated curve. In addition, this smoothing parameter must verify that it tends to zero "slowly", i.e., $h \to 0, nh \to \infty$ to ensure that $\hat{f}$ tends to the true density f And on the other hand the kernel function K, which there are different types to use and whose selection is usually set by the researcher. More details about kernels are discussed in [2].

To assign a density to the value x_0 in (2) where we want to calculate $\hat{f}(x_0)$ we open an interval of length h centered on

DOI: https://doi.org/10.30564/ese.v3i1.3368

x_0. The more data in that interval $\hat{f}(x_0)$ is higher, but it is not a value directly proportional to the number of data in that interval, rather these are weighted as a function of distance, by the function $K\left(\frac{x-X_i}{h}\right)$. The way $\hat{f}(x_0)$ is constructed ensures that the final curve obtained is a smooth continuous curve.

It is important to note that, this estimation technique is widely used due to its properties, including an asymptotically optimal kernel estimate in L_1 and strong convergence in L_1 for kernel density estimates, which can be appreciated in more detail in [3,4].

3.2 Considerations in the Kernel Density Estimation Implementation

Having into account that GMFM is a process with ergodic properties and time averages and spatial averages converges, if the process evolves enough time, we can analyze the traffic rates. The traffic speeds values can be considered as a random variable with distribution π. Hence we can estimate its probability density function (p.d.f.) by means of kernel methods.

As mentioned above, the window size parameter h is an important aspect of these techniques, its general behavior is that the larger the window, the smoother the p.d.f. estimated. Therefore, a "large" h could lead to joining nearby peaks into one and drawing incorrect conclusions, but a "small" h could show too many peaks leading to false highs.

Using an appropriate window size, we find that this p.d.f. is multimodal, and based on the shape of this distribution we will estimate the ranges and average rates of dispatch. Finally, we estimate the probability π_i of each state, with the area under the p.d.f. within each range. With this information we reconstruct the modulating chain by assigning each instant to the corresponding state, and compute the estimators presented in [1]. This can be done because both, spatial and time behavior converges in the GMFM. Analyzing the minima and maxima we estimate both the elements of H and the range of values associated with each state of the modulating chain.

3.3 Infinitesimal Generator Estimation

To obtain $\hat{Q}$ there are different approaches, the method used in this work is to recover the state of the modulating chain by comparing the trace with the estimated ranges from $\hat{\pi}$. Once the modulating chain is reconstructed, we can estimate the elements in $\hat{Q}$ by the ratio between the number of transitions from state i to state j and the time that the chains remains at time i.

4. Simulation and Numerical Results

In this section we will carry out the analysis with simulated traffic traces generated by simulations to perform the kernel estimations. Simulations were performed in Python 3.7 using sklearn.neighbors library [7] and codes can be provided by asking to the authors.

4.1 Parameters to Simulation

Several traffic simulations were performed according to the presented model were performed where the modulating Markov chain has $k = 9$ states and each state is associated with a data transfer rate interval shown the table below.

To design the infinitesimal generator of the chain we took into account some considerations like that, it is desired that the usual state be that of the highest transfer rate available in the transmission channel, so the most probable state is the ninth. It is also more common in the actual behavior of a transmission channel to jump from one state to the adjacent ones, to the maximum transfer rate, or to the minimum rate of transfer, so

State	Transfer speed (Mbps)
1	(0, 1024]
2	(1024, 2048]
3	(2048, 3072]
4	(3072, 4096]
5	(4096, 5120]
6	(5120, 6144]
7	(6144, 7168]
8	(7168, 8292]
9	(8292, 10240]

$$Q = \begin{pmatrix} -7 & 2 & 0.13 & 0.13 & 0.13 & 0.13 & 0.13 & 0.13 & 4.22 \\ 2 & -7 & 2 & 0.13 & 0.13 & 0.13 & 0.13 & 0.13 & 2.35 \\ 1 & 2 & -7 & 2 & 0.13 & 0.13 & 0.13 & 0.13 & 1.48 \\ 1 & 0.125 & 2 & -7 & 2 & 0.13 & 0.13 & 0.13 & 1.485 \\ 1 & 0.125 & 0.13 & 2 & -7 & 2 & 0.13 & 0.13 & 1.485 \\ 1 & 0.125 & 0.13 & 0.13 & 2 & -7 & 2 & 0.13 & 1.485 \\ 1 & 0.125 & 0.13 & 0.13 & 0.13 & 2 & -7 & 2 & 1.485 \\ 2 & 0.20 & 0.20 & 0.20 & 0.20 & 0.20 & 0.20 & -8 & 4.8 \\ 2 & 0.30 & 0.30 & 0.30 & 0.30 & 0.30 & 0.30 & 5 & -8.8 \end{pmatrix}$$

Within each of these intervals, how much is actually dispatched is drawn by means of a probability distribution, in the first case by a Gaussian distribution and in the second case by an Exponential.

The simulated trace is a succession of pairs $(v_i; t_i)$, with i from 0 to 20000, where v_i is the transfer speed, t_i is the moment when the chain jumps to another state and 20000 is the number of jumps in the chain, so the link transfers at the speed v_i while $t_{i-1} < t < t_i$.

4.2 Estimations from Traces

4.2.1 Gaussian Distribution Case

Parameters for the modulating chain were introduced in section 4.1. In this case we consider that within each interval the dispatched is drawn from a Gaussian

DOI: https://doi.org/10.30564/ese.v3i1.3368

distribution centered to its midpoint and deviation equal to one sixth of the length of the interval.

Figure 1 shows the first 150 jumps of a trace as an example, to visualize with which data we are going to work.

For each simulated trace we estimate the EB through the following steps:

(1) Apply a Gaussian kernel to all $v_i, 0 \leq i \leq 20000$,with h = 200, to obtain $\widehat{\pi}(x)$ for $0 < x < 10240$. This is possible because GMFM are ergodic, and time and spaces averages converge. See Figure 2.

(2) Find minima for $\widehat{\pi}(x)$. These minima are an estimate for the extremes of the dispatch ranges, which in turn allow us to determine the state of the modulating chain. As Gaussian distribution is symmetric, we determine rate averages using the estimated rank middle points. Finally, area under $\widehat{\pi}(x)$ between two consecutive minima estimates $\widehat{\pi}_i$.

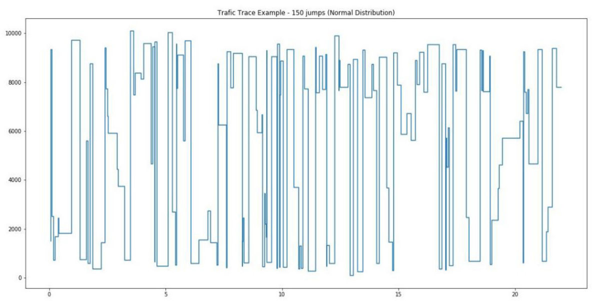


Figure 1. Gaussian distribution traces examples.

(3) Go through the trace comparing each vi with the rank estimated to assign the corresponding state, to obtain the estimated chain $(\hat{c}_i, t_i), 0 \leq i \leq 20000$.

(4) Estimate infinitesimal generator from $(\hat{c}_i, t_i)$ where t_i are cumulative so first order difference of t_i gives permanence time in state c_i.

(5) Calculate the estimated EB with $\widehat{H}$, $\widehat{\pi}$,and $\widehat{Q}$, as in (1).

The choice over the value of 200 for the width of the kernel window is somewhat heuristically determined. Smaller values generate an estimated trace with much more local maxima, making this search difficult to automate.

Figure 2 shows the theoretical and estimated density; Table 1 shows the estimated ranges of dispatch and Table 2 the estimated average dispatch rates.

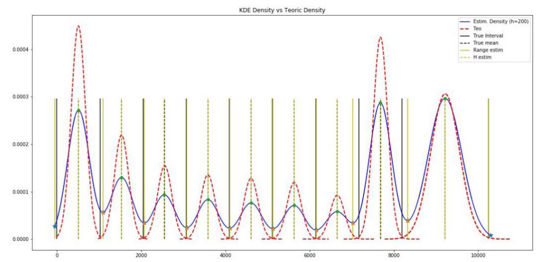


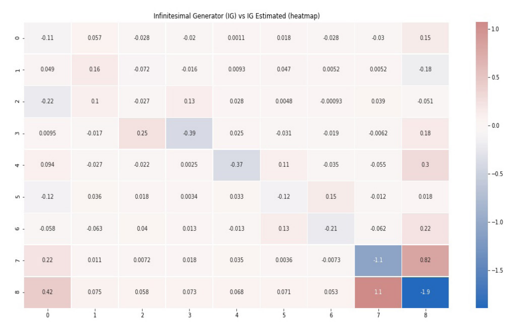
Figure 2. Theoretical and estimated density, using Kernel Estimation techniques.

The estimation of the infinitesimal generator is as follows

$$Q=\begin{pmatrix} -6.8935 & 1.9432 & 0.1578 & 0.1501 & 0.1289 & 0.1116 & 0.1578 & 0.1597 & 0.1597 \\ 1.9513 & -7.1605 & 2.072 & 0.1456 & 0.1207 & 0.0832 & 0.1248 & 0.1248 & 2.5297 \\ 1.2182 & 1.8956 & -6.9733 & 1.8728 & 0.1025 & 0.1252 & 0.1309 & 0.0911 & 1.5313 \\ 0.9905 & 0.1424 & 1.7519 & -6.6115 & 1.9748 & 0.161 & 0.1486 & 0.1362 & 1.3062 \\ 0.9061 & 0.1521 & 0.1521 & 1.9975 & -6.634 & 1.885 & 0.1654 & 0.1852 & 1.1839 \\ 1.1248 & 0.0894 & 0.1117 & 0.1266 & 1.9665 & -6.8754 & 1.8474 & 0.1415 & 1.4675 \\ 1.0581 & 0.1883 & 0.0897 & 0.1166 & 0.1435 & 1.8651 & -6.7879 & 2.0624 & 1.2643 \\ 1.7767 & 0.1891 & 0.1928 & 0.1818 & 0.1655 & 0.1964 & 0.2073 & -6.9357 & 3.977 \\ 1.5757 & 0.2246 & 0.2421 & 0.2269 & 0.2316 & 0.2293 & 0.2468 & 3.9292 & -6.9074 \end{pmatrix}.$$

The heat map for the error in the estimation of infinitesimal generator shown in the Table 3, help us to evaluate the performance of the estimator.

Table 3. Heatmap for error estimation in the infinitesimal generator.



In addition, the confusion matrix M that we show below,

Table 1. Theoretical and estimated ranges of dispatch using Kernel Estimation techniques.

Theoretical range	0	1024	2048	3072	4096	5120	6144	7168	8192	10240
Estimated range	0	1091.74	2076.63	3083.74	4099.60	5125.89	6152.85	7020.94	8321.57	10240
Error	0	67.74	28.63	11.74	3.6	5.89	8.85	147.06	129.57	0

Table 2. Theoretical and estimated average dispatch rates.

Theoretical Average rates	512	1536	2560	3584	4608	5632	6656	7680	9216
Estimated average rates	516.16	1537.06	2548.88	3585.94	4607.19	5629.44	6653.37	7678.99	9213.89
Error	4.16	1.06	11.12	1.94	0.81	2.56	2.63	1.01	2.11

DOI: https://doi.org/10.30564/ese.v3i1.3368

allow us to evaluate the performance in the estimation of the states.

$$M = \begin{pmatrix} 3521 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 375 & 1317 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 260 & 952 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 247 & 866 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 224 & 793 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 197 & 739 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 176 & 578 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 789 & 2973 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1330 & 4662 \end{pmatrix}.$$

Rows are the actual states and columns are the predicted or estimated states. For example, the 260 in matrix M at row 3, column 2 indicates that two hundred and sixty times the chain was in state 3 but was estimated to be in state 2.

Figure 3 shows the comparison of the estimated EB for a trace with the theoretical value.

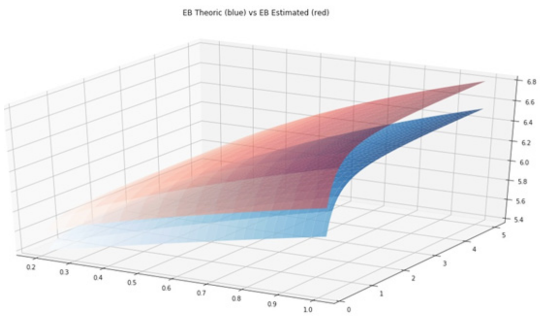


Figure 3. Theoretical bandwidth (blue) vs. Estimated bandwidth (red)

We can see that estimated EB is always above real one. This is not a problem because overflow probability and other QoS estimation, are conservative calculated, i.e., real probability is less than the estimated.

4.2.2 Exponential Distribution Case

Parameters for the modulating chain were introduced in section 4.1, in this case we consider that within each interval the dispatched is drawn from an exponential distribution with mean value is 100 MB from interval origin. If the interval is [1024, 2048), the exponential mean parameter is at 1124 MB.

Figure 4 shows the first 150 jumps of a trace as an example, to visualize with which data we are going to work.

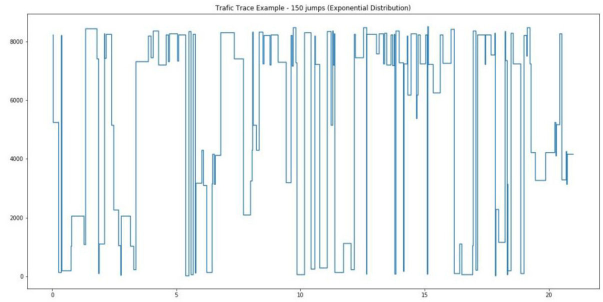


Figure 4. Exponential distribution traces examples.

For each simulated trace we estimate the EB through the following steps:

(1) Apply a Gaussian kernel to all v_i, $0 \le i \le 20000$, with $h = 20$, to obtain $\hat{\pi}(x)$ for $0 < x < 1024$. This is possible because GMFM are ergodic, and time and spaces averages converge. See Figure 5.

(2) Find maxima for $\hat{\pi}(x)$. These maxima are an estimate for the extremes of the dispatch ranges, which in turn allow us to determine the state of the modulating chain. Exponential distribution is not symmetric. So, we determine rate averages empirically calculating mean value of all the times that dispatch was within interval determinate by this maximum. Finally, area under $\hat{\pi}(x)$ between two consecutive maxima estimates $\hat{\pi}_i$.

(3) Go through the trace comparing each vi with the rank estimated to assign the corresponding state, to obtain the estimated chain $(\hat{c}_i, t_i)$ $0 \le i \le 20000$.

(4) Estimate infinitesimal generator from $(\hat{c}_i, t_i)$ where t_i are cumulative so first order difference of t_i gives permanence time in state c_i.

(5) Calculate the estimated EB with $\hat{H}$, $\hat{\pi}$ and $\hat{Q}$ as in (1).

The choice over the value of 20 for the width of the kernel window is somewhat heuristically determined.

Table 4. Theoretical and estimated ranges of dispatch using Kernel Estimation techniques.

Theoretical range	0	1024	2048	3072	4096	5120	6144	7168	8192	10240
Estimated range	0	1052.14	2074.48	3101.50	4124.17	5155.21	6175.87	7198.54	8221.54	10240
Error	0	28.14	26.48	29.50	28.17	35.21	31.87	30.54	29.54	0

Table 5. Theoretical and estimated average dispatch rates.

Average theoretical rates	100	1124	2148	3172	4196	5220	6244	7268	8292
Estimated average rates	200.93	1318.37	2389.62	3418.37	4461.25	5459.18	6839.68	7613.36	8321.96
Error	100.93	194.37	241.62	246.37	265.25	239.18	595.68	345.36	29.96

 DOI: https://doi.org/10.30564/ese.v3i1.3368

Bigger values generate an estimated trace with less local maxima, but more “normalized”, so estimation of area under the curve fall far from actual values.

Figure 5 shows the theoretical and estimated density; Table 4 shows the estimated ranges of dispatch and Table 5 the estimated average dispatch rates.

In this case the estimation of the infinitesimal generator is as follows

$$Q=\begin{pmatrix} -6.9393 & 1.4441 & 0.2797 & 0.1416 & 0.1381 & 0.1206 & 0.1311 & 1.0875 & 2.9145 \\ 1.8619 & -6.7947 & 1.3564 & 0.2991 & 0.1053 & 0.1559 & 0.1137 & 0.6445 & 1.5418 \\ 1.4328 & 1.1773 & -6.8974 & 1.3106 & 0.4276 & 0.1499 & 0.1388 & 0.4887 & 1.0329 \\ 0.9056 & 0.4787 & 1.3261 & -6.8505 & 1.2356 & 0.5757 & 0.1423 & 0.4334 & 1.0286 \\ 1.2968 & 0.153 & 0.4881 & 1.2312 & -7.2344 & 1.2822 & 0.4153 & 0.4954 & 1.0928 \\ 1.0977 & 0.1432 & 0.1511 & 0.5648 & 1.2409 & -7.1909 & 1.4477 & 0.692 & 0.9705 \\ 1.4889 & 0.1396 & 0.2047 & 0.1535 & 0.3955 & 0.6281 & -7.1049 & 1.2888 & 2.4288 \\ 1.7487 & 0.1901 & 0.218 & 0.1967 & 0.195 & 0.2065 & 0.5425 & -7.0048 & 2.0405 \\ 1.6385 & 0.2325 & 0.2515 & 0.2625 & 0.2436 & 0.2119 & 1.1498 & 2.997 & -6.9872 \end{pmatrix}$$

The heat map for the error in the estimation of infinitesimal generator shown in the Table 6, help us to evaluate the performance of the estimator.

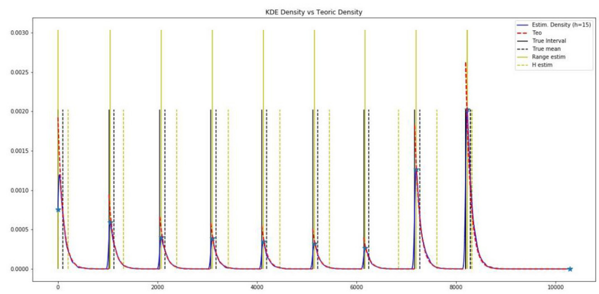


Figure 5. Theoretical and estimated density, using Kernel Estimation techniques.

Table 6. Heatmap for error estimation in the infinitesimal generator

The confusion matrix M that we show below, allow us again to evaluate the performance in the estimation of the states

$$M=\begin{pmatrix} 3534 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 426 & 1324 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 290 & 953 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 289 & 803 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 256 & 733 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 260 & 700 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 204 & 574 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 953 & 2795 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1479 & 4418 \end{pmatrix}.$$

Figure 6 shows the comparison of the estimated EB for a trace with the theoretical value.

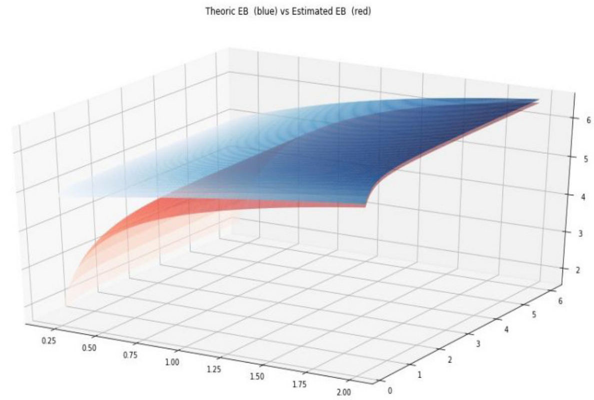


Figure 6. Theoretical bandwidth (blue) vs. Estimated bandwidth (red).

We can see that estimated EB is always under real one. This behavior is opposite than the Gaussian case, but this is also not a problem because convergence is fast enough, in order that overflow probability and other QoS estimation, are near enough to its real values.

5. Conclusions

In this paper we have proposed a non-parametric methodology to estimate effective bandwidths from traffic traces of a GMFM source with expected properties. These results allow us to estimate the effective bandwidth from traffic traces with very little prior knowledge of the GMFM. Of course, we pay for the versatility of this method with a slightly larger estimation gap.

The estimation involves the calculation of the maxima and minima of the estimated density function. This process is easy for the human eye but it is perhaps the most complicated part to implement computationally because noise can generate spurious maxima or minima.

Numerical examples of simulated traces were presented showing the results obtained. The estimation process worked much better in the Gaussian distribution case than in the Exponential one, which can be seen in both the heat map and the confusion matrix presented for each case.

It is expected to extend the statistical calculation to the more realistic case where the number of dispatch classes is not known, distributions are not of the same family, and also where the supports of each probability law have bigger intersection, in order to develop the estimation to real data scenarios.

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