

Journal of Architectural Environment & Structural Engineering Research <https://ojs.bilpublishing.com/index.php/jaeser>

ARTICLE Failure Evaluation of Reinforced Concrete Beams Using Damage Mechanics and Classical Laminate Theory

José Mário Feitosa Lim[a](https://orcid.org/0000-0002-2294-3304) Geraldo José Belmonte dos Santo[s](https://orcid.org/0000-0001-6726-7330) Paulo Roberto Lopes

Lima[*](https://orcid.org/0000-0003-2937-9520)

State University of Feira de Santana, Technology Department, Feira de Santana, Bahia, 44030-900, Brazil

ARTICLE INFO ARTICLE IN A RESERVE A LIMIT OF A RESERVE A LIMIT OF A

Article history Received: 30 August 2022 Revised: 16 September 2022 Accepted: 20 October 2022 Published Online: 26 October 2022

Keywords: Reinforced concrete Damage mechanics Finite element method Laminate theory

The prediction of the behavior of reinforced concrete beams under bending is essential for the perfect design of these elements. Usually, the classical models do not incorporate the physical nonlinear behavior of concrete under tension and compression, which can underestimate the deformations in the structural element under short and long-term loads. In the present work, a variational formulation based on the Finite Element Method is presented to predict the flexural behavior of reinforced concrete beams. The physical nonlinearity due cracking of concrete is considered by utilization of damage concept in the definition of constitutive models, and the lamination theory it is used in discretization of section cross of beams. In the layered approach, the reinforced concrete element is formulated as a laminated composite that consists of thin layers, of concrete or steel that has been modeled as elasticperfectly plastic material. The comparison of numerical load-displacement results with experimental results found in the literature demonstrates a good approximation of the model and validates the application of the damage model in the Classical Laminate Theory to predict mechanical failure of reinforced concrete beam. The results obtained by the numerical model indicated a variation in the stress–strain behavior of each beam, while for under-reinforced beams, the compressive stresses did not reach the peak stress but the stress–strain behavior was observed in the nonlinear regime at failure, for the other beams, the concrete had reached its ultimate strain, and the beam's neutral axis was close to the centroid of the cross-section.

1. Introduction

The nonlinear numerical analysis of reinforced concrete structures has been implemented to predict both the reduction in stiffness with the increase in deformations, as well as the mechanism and process of failure [1]. The incorporation of nonlinear stress–strain models under tension and/or compression $[2-4]$ has changed the constitutive equations of concrete.

Concrete is a cement-based composite material whose

Paulo Roberto Lopes Lima,

DOI:<https://doi.org/10.30564/jaeser.v5i4.5028>

Copyright © 2022 by the author(s). Published by Bilingual Publishing Co. This is an open access article under the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0) License. (https://creativecommons.org/licenses/by-nc/4.0/).

^{*}Corresponding Author:

State University of Feira de Santana, Technology Department, Feira de Santana, Bahia, 44030-900, Brazil; *Email: lima.prl@pq.cnpq.br*

is divided into several layers [2,16]

mechanical properties depend on the constituents and vere compared with interfaces between them. Its behavior is defined by the the literature. pre-existence of pores, voids, inclusions, and microcracks prior to loading, which induces: (i) a post-cracking be-
alinearly and Numerical Modhavior of strain softening; (ii) progressive deterioration of eling the mechanical properties; (iii) volumetric expansion; (iv) induced anisotropy; (v) asymmetry in response to traction **2.1 Materials Moo** and compression; (iv) considered fragile in traction and quasi-ductile in compression. In contrast, reinforced con-
The model proportional method is $\frac{1}{\sqrt{N}}$. results allows in compression, in comparison, community compared with the cemen-
crete uses steel reinforcements embedded in the cementitious matrix to increase the strength and stiffness of the composite, primarily in the tension regions. Reinforced concrete and degrades owing to distributed interocracking caused concrete exhibits an initially linear elastic behavior with a progressive increase in loading that progress to a nonlinear inelastic behavior induced by crack propagation when an equivalent strain and concrete crushing or steel yielding. Mathematical value, ϵ_{d0} , corresponding and concrete modeling of the nonlinear inelastic behavior of concrete,
local state and is given without considering creep, is typically based on plasticity local state and is given by: without considering creep, is typicarly based on plasticity
theory, continuous damage theory, fracture mechanics, or $\epsilon = \sqrt{\epsilon_0^2 + \epsilon_0^2}$ $\frac{1}{2}$ combination of these $\frac{[5-12]}{2}$. $\frac{1}{2}$ continuate expansion, (1)
mmetry in response to traction 2.1 Materials Modelling **2.1 Materials Modelling** $\frac{d}{dt}$ and the adoption of lamination of lamination, whereinthe structural elements $\frac{d}{dt}$ $\text{in } \text{s}$ in terms of $\text{in } \text{t}$ and reinforcement bars are modeled separately using the modeled separately using the elements. And the elements of $\frac{1}{2}$

The isotropic Mazar damage model [13] allows the continuous representation of the structural model even after princ concrete cracking has generated good results in the mod-
 ϵ_i , if ϵ_j eling of reinforced concrete structures ^[14,15]. This model $\langle \varepsilon_i \rangle_+ = \frac{1}{2} (\varepsilon_i + |\varepsilon_i|) = \begin{cases} \varepsilon_i, t_j & \varepsilon_i > 0 \\ 0, if \varepsilon_i \le 0 \end{cases}$ (2) uses theories based on the mechanics of continuous damage that define the constitutive laws of concrete. Only one internal variable is required to apply this model and its dent is a problem. meethal variable is required to apply this model and its
evolution law is easily obtained by performing tensile and compression tests on the material. d by performing tensile and $\overline{}$ li Internal variable is required to apply this model and its μ phase, when the straighteen low is easily obtained by performing tensils and evolution law is easily obtained by performing tensile and limit $(\varepsilon > \varepsilon_{d0})$, the

crete structures, the most common model for numerical
Thus the uniaxial stress–strain behavior of concrete analysis has been the use of the finite element method, can wherein the concrete and reinforcement bars are mod-
 $(E_0 \varepsilon, \varepsilon)$ $\varepsilon \le \varepsilon_{d0}$ eled separately using two different types of elements. An $\sigma = \begin{cases} E_0 & \varepsilon \le \varepsilon_{d0} \\ (1-D)E_0\varepsilon, & \varepsilon > \varepsilon_{d0} \end{cases}$ additional approach has been used with the adoption of lamination, wherein the structural element is divided into In Equation (3), several layers ^[2,16]. Based on the classical laminate theory, this model associates a specific type of material with each $\frac{\text{prety languageu}}{\text{real}}$, as layer of the beam and considers the perfect adhesion be-
 $D = \alpha_T D_T + \alpha_C D_C$ (4) tween the layers. By monitoring the stresses and strains in each layer, the commencement of cracking in the concrete $\sum_i < \varepsilon_{T_i} > +$ (5) and the yielding of the reinforcement can be identified, thus resulting in a more realistic evaluation of the behavior of the structural elements of reinforced concrete. reory tween 0 (when $\varepsilon \le \varepsilon_{d0}$) and 1 (when several layers [2,16]. Based on the classical laminate theory,

> The objective of this study is to assess the effectiveness of damage mechanics and classical lamination $D_{\pi}(\tilde{s}) = 1 - \frac{\varepsilon_{d0}(1-A_T)}{2} - \frac{A_T}{2}$ theory in the failure prediction of reinforced concrete beams. For this purpose, a variational formulation model was developed based on such theories and the principle of virtual work. Subsequently, it was applied using the finite element method (FEM) and the obtained results

d on the constituents and were compared with the experimental results reported in the literature. $\mathbf{r} = \mathbf{r} \cdot \mathbf{r}$ realistic evaluation of the behavior of the structural elements of reinforced concrete. the layers. By monitoring the stresses and strains in each layer, the commencement of cracking in additional approach has been used with the adoption of lamination, wherein the structural element additional approach has been used with the adoption of lamination, wherein the structural element $t_{\rm max}$ the complete and the emperiment concrete ϵ is the intervals of the results ϵ realistic evaluation of the structural elements of the structural elements of reinforced concrete.

. Based on the classical laminosity of the classical laminosity \mathcal{A}

is divided into several layers $\overline{2}$

the layers. By monitoring the stresses and strains in each layer, the commencement of cracking in

the layers. By monitoring the stresses and strains in each layer, the commencement of cracking in

2. Problem Formulation and Numerical Modeling

s, inclusions, and microcracks
luces: (i) a post-cracking be- 2. Problem Formulation and Numerical Mod**eling** T_{total} of θ assess the effective of θ assess the effectiveness of θ and $\$ realing the behavior of the structural elements of \mathbf{r} reinforced concrete. $\frac{1}{\pi}$ in the failure prediction of reinforced concrete beams. For this purpose, $\frac{1}{\pi}$ a variation model was developed based on such the principle on such the principle of virtual the principle of virtual of virtual the principle of virtual the principle of virtual of virtual the principle of virtual of vir model for numerical analysis has been the use of the finite element method, wherein the concrete

classical lamination theory in the failure prediction of reinforced concrete beams. For this purpose, results were compared with the experimental reported in the literature. The literature reported in the literature.

red fragile in traction and
n contrast reinforced con
The model proposed by Mazars^[13] is based on expern. In contrast, reinforced con-
ents embedded in the cemen-
imental evidence observed in the behavior of concrete estength and stiffness of the under uniaxial tension and compression, wherein the mae strength and stiffness of the
terrail degrades owing to distributed microcracking caused
terrail degrades owing to distributed microcracking caused ion regions. Reinforced
are lastic behavior with the stresses. In this model, the damage is reprey linear elastic behavior with $\frac{1}{2}$ electron in the second in the second sense of the second sense of a non-
ading that progress to a noning that progress to a non-
ced by crack propagation when an equivalent strain measure, $\tilde{\epsilon}$, exceeds a threshold duced by crack propagation
steel yielding. Mathematical value, ε_{d0} , corresponding to the tensile strength of the steel yielding. Mathematical
nelastic behavior of concrete, $\frac{1}{2}$ concrete. The equivalent strain $\tilde{\epsilon}$ represents the elongation astic behavior of concrete,

voically based on plasticity local state and is given by: in contrast, reinforced con-
The model proposed by Mazars^[13] is based on experon regions. Reinforced $\frac{1}{2}$ theories and the principle of virtual theories and the principle of virtua y linear elastic behavior with by tensile stresses. In this model, the damage is repreidered fragile in traction and
n. In contrast, reinforced con-
The model proposed by Mazars ^[13] is based on experduced by crack propagation when an equivalent strain measure, $\tilde{\epsilon}$, exceeds a threshold

theory, fracture mechanics, or
$$
\tilde{\varepsilon} = \sqrt{&\ \varepsilon_1 >_+^2 + &\ \varepsilon_2 >_+^2 + &\ \varepsilon_3 >_+^2} \tag{1}
$$

e model ^[13] allows the con-
where $\langle \varepsilon_i \rangle_+$ is the positive part of the elongation in the e structural model even after principal direction *i* and is defined as \mathfrak{g} where ϵ is the principal direction in the principal direction in the principal direction ϵ and is defined as age model ¹³³ allows the con-
where $\langle \varepsilon_i \rangle$ is the positive part of the elongation in the

allts in the mod-

\n
$$
\begin{aligned}\n &\text{allts in the mod-}\\
 &< \varepsilon_i > \pm \frac{1}{2} (\varepsilon_i + |\varepsilon_i|) = \begin{cases}\n\varepsilon_i, & \text{if } \varepsilon_i > 0 \\
0, & \text{if } \varepsilon_i \leq 0\n\end{cases}\n\end{aligned}\n\tag{2}
$$

compression tests on the material.
In terms of discrete representation of reinforced con-
damaged nics of continuous dam-
In the pre-cracking phase ($\varepsilon \le \varepsilon_{d0}$), the concrete exhib-
us of concrete Only one laws of concrete. Only one its linear elastic behavior. Whereas in the post-cracking μ apply this model and its
hase, when the strain is greater than the elastic strain
how performing tensile and $\lim_{a \to a_0} (c > c_0)$, the concrete exhibits hominear easite be-
havior when the initial elastic modulus E_0 is progressively na.
hihita non hanics of continuous dam-
laws of concrete Only one is the pre-cracking phase ($\varepsilon \le \varepsilon_{d0}$), the concrete exhib $t_c \geqslant \epsilon_{d0}$, the concrete extributs nonlinear elastic \mathbf{red} , \mathbf{ord} , damaged. explans of concrete. Only one its linear elastic behavior. Whereas in the post-cracking
t to apply this model and its forming tensile and $\lim_{k \to \infty} (\varepsilon > \varepsilon_{d0})$, the concrete exhibits nonlinear elastic be- α modulus α is progressively damaged. $d \quad \lim_{x \to 0}$ erforming tensile and $\lim_{\epsilon \to \epsilon_0}$ limit ($\epsilon > \epsilon_{d0}$), the concrete exhibits nonlinear elastic be--cracking phase ($\varepsilon \leq \varepsilon_{d0}$), the hereas in the $\lim_{\epsilon \to 0} (e^z - e_{a0})$, the concrete exhibits hominear classic be
havior when the initial elastic modulus F_a is progressively

frace *R* concrete *D* variable stress-strain behavior of concrete σ matrice depending by σ Thus, the uniaxial stress-strain behavior of concrete the element method, can be established by:
most have an used component included to the interior concrete.
Thus, the unitial stress-strain behavior of concrete

From this are from the following matrices:

\n
$$
\sigma = \begin{cases}\nE_o \, \varepsilon, & \varepsilon \leq \varepsilon_{d0} \\
(1-D) \, E_o \varepsilon, & \varepsilon > \varepsilon_{d0}\n\end{cases} \tag{3}
$$
\nwith the adoption of

Equation (3) the damage parameter D varies beelement is divided into the equation (5), the damage parameter *D* varies be-
lassical laminate theory, tween 0 (when $\varepsilon \le \varepsilon_{d0}$) and 1 (when the material is com- $\frac{a}{b}$ of $\frac{b}{c}$ the classical laminate theory, we we will be very and 1 (when the material is com-
fic type of material with each pletely damaged), as expressed by (see $^{[14,15]}$): $\frac{1}{2}$ In Equation (3), the damage parameter *D* varies be-
tween 0 (when $S \leq S(0)$ and 1 (when the meterial is com-In Equation (3), the damage parameter *D* varies be-

$$
D = \alpha_T D_T + \alpha_C D_C
$$
 (4)

with with $\frac{1}{2}$ or $\frac{$

$$
\alpha_T = \frac{\sum_i \langle \varepsilon_{T_i} \rangle_+}{\sum_i \langle \varepsilon_{T_i} \rangle_+ + \sum_i \langle \varepsilon_{C_i} \rangle_+} \tag{5}
$$

rate

\nrate

\n
$$
\alpha_{\mathcal{C}} = \frac{\sum_{i} < \varepsilon_{\mathcal{C}_{i}} > +}{\sum_{i} < \varepsilon_{\mathcal{T}_{i}} > + + \sum_{i} < \varepsilon_{\mathcal{C}_{i}} > +}
$$
\n(6)

$$
D_T(\tilde{\varepsilon}) = 1 - \frac{\varepsilon_{d0}(1 - A_T)}{\tilde{\varepsilon}} - \frac{A_T}{\exp\left[B_T(\tilde{\varepsilon} - \varepsilon_{d0})\right]}, \text{ and} \tag{7}
$$

$$
D_C(\tilde{\varepsilon}) = 1 - \frac{\varepsilon_{d0}(1 - A_C)}{\tilde{\varepsilon}} - \frac{A_C}{\exp[B_C(\tilde{\varepsilon} - \varepsilon_{d0})]} \tag{8}
$$

it was applied using the where $\alpha_T + \alpha_C = 1$, ε_{T_i} and ε_{C_i} are the components of the
and the obtained results principal strains determined by the positive and perstine (M) and the obtained results principal strains determined by the positive and negative positive and negative parts, respectively. The values $\mathcal{P}^{\text{max}}_{\text{max}}$

Τ

, (a) and (a)

, (a) and (a)

parts, respectively. The values A_T , B_T , A_C , B_C and ε_{d0} are the damage that the experimental parameters obtained from the material by Equation by the experimental parameters obtained from the material F tests, the reinforcing steel bars, the linear elastic behavior between the stress and tests. positive and negative parts, respectively. The values of α and α are the experimental strain positive and negative parts, respectively. The values of \mathcal{L} and \mathcal{L} and \mathcal{L} are the experimental \mathcal{L} $\frac{1}{\sqrt{2}}$ where the components of the components of the components of the principal strains determined by the principal strains determined by the principal strains of the principal strains of the principal strains of the principal s where the components are the material strains determined by the principal strains determined by the principal strains determined by the principal strains of the principal strains of the principal strains of the principal s eters obtained from the material by E \mathbf{S}

Ξ

Ξ

where $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ are the principal strains determined by the principal strains

where $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$ are the components of the principal strains determined by the principal strains

(− − ∞0)

(− − ∞0)

− ^ߝ

For the remoteing steel bars, the linear elastic behav-
ior between the stress and strain before yield deformation The classical
is accounted to $\tau = E$, i.e., ζ , ζ , where E is the abotic and ζ , and in the is assumed to be $\sigma_s = E_s \varepsilon$, if $\varepsilon < \varepsilon_y$, where E_s is the elastic only for laminates formed by orthotopic or $\frac{1}{2}$ is assumed to be $\frac{3}{2}$ and $\frac{2}{2}$ behavior. The step $\frac{1}{2}$ is the exhibit assumed to be constant, $\sigma_s = f_y$, where f_y is the yielding extension of the classical the assumed to be constant, $\sigma_s - f_y$, where f_y is the yielding extension of the classical the
stress of the steel and exhibits linearly elastic-perfectly volving materials subject to d tests. strain relations
For the reinforcing steel bars, the linear elastic behav-
ior between the stress and strain before yield deformation. modulus of the steel. After yielding $\varepsilon \ge \varepsilon_y$, the stress is terials ^[16]. However, this theory can be c plastic behavior. steel bars, the linear elastic behavior between the stress and strain before yield deformation The classical laminate theory is typically developed $\mathbf{f}_\mathbf{r}$, as shown in Figure 1. In this study, the cross section of the beams was discretized in NC layers of thickness *hk* In this study, the cross section of the beams wasdiscretized in NC layers of thickness *hk* modulus of the steel. After yielding $\varepsilon \ge \varepsilon_y$, the stress is the modulus of the steel. After yielding $\varepsilon \ge \varepsilon_y$, the stress is tests.
For the reinforcing steel bars, the linear elastic behavior het
wear the stress and strain before yield deformation modulus of the steel. After yielding $\varepsilon \le \varepsilon_y$, the stress is
assumed to be constant, $\sigma_s = f_y$, where f_y is the yielding

In this study, the cross section of the beams was discre-
2.2 Internal Stress Resu tized in NC layers of thickness h_k (k=1,..., NC), as shown **Section** in Figure 1. $\text{if } \text{if } x \in \mathbb{R}^n \text{ and } \text{if } x \in \mathbb{R}^n, \text{if } x \$ μ m ans study, the cross section of the material tests. \mathbf{r}_e 1 In this study, the cross section of the beams was discre- $\frac{2.2 \text{ Internal Stress Res}}{2.2 \text{ Internal Stress Res}}$ In this study, the cross section of the beams was discre $r_{\rm e}$ 1 tized in NC layers of thickness h_k (k=1,..., NC), as shown **Section**
in Figure 1 \sum regular steel bars, the linear elastic behavior between the stress and st

 $\mathbf{a}^{\mathcal{A}}$ σy explored in the contract of th
Contract of the contract of th $\frac{1}{2}$ $\frac{1}{2}$ $\ddot{}$ Figure 1. Discretization of the laminated beam where N denotes **Figure 1.** Discretization of the laminated beam
where M denotes the norm parameters obtained from the material tests.

t i. *Diserctizatio*
l laminate theory Classical laminate theory establishes that the laminae moment; b denotes the width of the section; and h denotes that form the laminate are in a plane stress state. In this its height. blishes that form the tammate are in a plane stress state. In this artistic lieght.
context, and based on the generalized Hooke's law for For a lam context, and based on the generalized Hooke's law for the For a laminated cross section
homogeneous and isotropic materials, the following relation of these internal s ween suesses and su tionship between stresses and strains in each layer of the by adding the contribution f \overline{a} sical laminate theory blishes that the laminae moment where the h c $\frac{15}{10}$ valid \mathbb{R}^2 . \mathbb{R}^n $\cos s$, d \overline{a} sical laminate theory εχει της κατά της και το κατά
Στην κατά της και το Classical laminate theory establishes that the laminae
that form the laminate property of the set of the laminate theory is a plane state of the laminate in the laminate in the state of the state of the state of the state where the h $\overline{}$ $\cos s$, $\cos s$ and 101
Ingles context, and ba v a $\frac{1}{2}$ and $\frac{1}{2}$ are $\frac{1}{2}$ and $\frac{1}{2}$ a $\frac{100}{2}$ laminate is vali σy $\ddot{}$ ext. and based on the ur
. context, and based on the generalized Hooke's law for Fo p betwe
to is yol: laminate is valid [16]. Some cases, different materials are laminate is valid [16] $\frac{1}{100}$ is the following relationship between strains in each layer of the $\frac{1}{100}$ adding the contribution in p_{in} is the generalized on the generalized based of the generalized Hooke's law for $\frac{1}{2}$ tionship between stresses and strains in each layer of the by adding the contribution fr context, and based on the generalized Hooke's law for For a laminated cross section **Figure 1.** Discretization of the late of the late of the laminated beams of the late of the laminated beams of the late of is other following relationship between stresses and yimogeneous and isotropic materials, the following rela- evaluation. Incore and isotropic materials, the following real evaluation of these international μ_{O} (k). The independent in Figure 1.1. In this study, the complete internation of the beams was discretized in NC layers **help** in NC layers **help** in NC layers **h** tionship between stresses and strains in each layer of the by adding the results. In this study of the benefit in the beams was discretized in NC layers of the beams of the beams of this is a h homogeneous and *isotropic* materials, the following relamaterials, the ronowing real evaluation of these interests of the resultants. and *h* cach hay control b section; and *h* denotes its height.

$$
\begin{Bmatrix}\n\sigma_x \\
\sigma_y \\
\tau_{xy}\n\end{Bmatrix} = \begin{bmatrix}\nQ_{11}^{(k)} & Q_{12}^{(k)} & 0 \\
Q_{12}^{(k)} & Q_{22}^{(k)} & 0 \\
0 & 0 & Q_{66}^{(k)}\n\end{bmatrix} \begin{Bmatrix}\n\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}\n\end{Bmatrix}
$$
\nment of the addition, we obtain Equations (16) and (17), which are compact expressions of the resultants.

\n(16)

\n(17)

\n(18)

\n(19)

\n(19)

\n(10)

\n(19)

\n(10)

\n(11)

\n(12)

\n(13)

\n(14)

\n(15)

\n(16)

\n(17)

\n(18)

\n(19)

\n(10)

\n(11)

\n(12)

\n(13)

\n(14)

\n(15)

\n(16)

\n(17)

\n(18)

\n(19)

\n(10)

\n(11)

\n(12)

\n(13)

\n(14)

\n(15)

\n(16)

\n(19)

\n(10)

\n(11)

\n(12)

\n(13)

\n(14)

\n(15)

\n(16)

\n(17)

\n(19)

\n(19)

\n(10)

\n(11)

\n(12)

\n(13)

\n(14)

\n(15)

\n(16)

\n(17)

\n(19)

\n(19)

\n(10)

\n(11)

\n(12)

\n(13)

\n(14)

\n(15)

\n(16)

\n(19)

\n(10)

\n(11)

\n(12)

\n(13)

\n(14)

\n(15)

\n(16)

\n(19)

\n(11)

\n(10)

\n(11)

\n

and $\frac{\partial u_1}{\partial x}$ = v_1 , v_2 , v_3 and v_6 are the the $M = B_{11} \frac{\partial u_0}{\partial x} - L$
al constants related to the engineering properties of $\frac{1}{1}$ oc \sim 1 where the quantities $Q_{11}^{(k)}$, $Q_{12}^{(k)}$, $Q_{22}^{(k)}$ and $Q_{66}^{(k)}$ are chanical constants related to the engineering prop where the quantities $Q_{11}^{(k)}$, $Q_{12}^{(k)}$, $Q_{22}^{(k)}$ and $Q_{66}^{(k)}$ are the me-
 $M =$ chanical constants related to the layer material, and the layer material, and the layer material, and the layer of the layer material, and the lawer material, and the lawer material, and the lawer material, and the lawer m engineering properties of the engine. where the quantities $Q_{11}^{(k)}$, $Q_{12}^{(k)}$, $Q_{22}^{(k)}$ and $Q_{66}^{(k)}$ are the me-
 $M = B_{11} \frac{\partial u_0}{\partial u} - D_{11} \frac{\partial^2 w_0}{\partial u^2}$ anical constants related to the engineering properties where the quantities $Q_{11}^{(k)}$, $Q_{12}^{(k)}$, $Q_{22}^{(k)}$ and $Q_{66}^{(k)}$ are the me-
 $M = B_{11} \frac{\partial u_0}{\partial t} - D_{11} \frac{\partial^2 w_0}{\partial t^2}$ chanical constants related to the engineering properties of where the quantities $Q_{11}^{(k)}$, $Q_{12}^{(k)}$, $Q_{22}^{(k)}$ and $Q_{66}^{(k)}$ are the me-
 $M = B_{11} \frac{\partial u_0}{\partial q} - D_{11} \frac{\partial^2 w_0}{\partial q^2}$ chanical constants related to the engineering properties where the quantities $Q_{11}^{(k)}$, $Q_{12}^{(k)}$, $Q_{22}^{(k)}$ and $Q_{66}^{(k)}$ are the me-
 $M = B_{11} \frac{\partial u_0}{\partial t} - D_{11} \frac{\partial^2 w_0}{\partial t^2}$ chanical constants related to the engineering properties are the mechanical constants related to the engineering properties of chanical constants related to the engineering properties of where the quantities $Q_k^{(k)}$ $Q_k^{(k)}$ α chanical constants related to the engineering properties of where the quantities $Q_{11}^{(k)}$, θ near constants related to the engineering properties of the layer material, as defined below. rties of $\frac{1}{1}$ \mathbf{v} .

where
\n
$$
Q_{11}^{(k)} = Q_{22}^{(k)} = \left(\frac{E}{1 - v^2}\right)^{(k)}
$$
\nwhere
\n
$$
Q_{11}^{(k)} = Q_{22}^{(k)} = \left(\frac{E}{1 - v^2}\right)^{(k)}
$$
\n(10) $A_{11} = b \sum_{k=1}^{NC} Q_{11}^{(k)}$

$$
Q_{12}^{(k)} = \left(\frac{\nu E}{1 - \nu^2}\right)^{(k)}, \text{ and}
$$
\n
$$
B_{11} = \frac{b}{2} \sum_{k=1}^{NC} Q_{11}^{(k)} \left(z_{k+1}^2 - z_k^2\right)
$$
\n
$$
D_{11} = \frac{b}{2} \sum_{k=1}^{NC} Q_{11}^{(k)} \left(z_{k+1}^2 - z_k^2\right)
$$
\n
$$
D_{11} = \frac{b}{2} \sum_{k=1}^{NC} Q_{11}^{(k)} \left(z_{k+1}^2 - z_k^2\right)
$$

$$
Q_{66}^{(k)} = \left(\frac{E}{2(1+v)}\right)^{(k)} = (G)^{(k)}
$$
\n
$$
D_{11} = \frac{b}{3} \sum_{k=1}^{NC} Q_{11}^{(k)} \left(z_{k+1}^3 - z_k^3\right)
$$
\nThe quantities A_{11} , B_{11} , an

where E and G represent the longitudinal and transverse
moduli of electroity of the meterial representively which moduli of elasticity of the material, respectively, which make up the layer of the laminate; and *v* is Poisson's ratio. τxy and ν is Poisson's ratio nate; and v is Poisson s ratio ia \cdot and ν is Poisson² reconsider which ινι
. \mathfrak{u} ₀ o rengituamar and transvers ι...
. $\mathop{\rm Im}\nolimits$ \mathbf{e}_1 and \mathbf{v} is Poisson's rationminate; and ν is Poisson's ration eyeried to intervely, which bending stiffness terial, respec
te: and v is P r's ration α **Principle Principle Principle** μ , and ν is tois.

In the formulation proposed here, the elastic modulus \overline{a} \mathfrak{c}_{11} , \mathfrak{c}_{12} , \mathfrak{c}_{22} $Q_{12}^{(k)}$, $Q_{22}^{(k)}$ and $Q_{66}^{(k)}$ incorporates $\bar{Q}_{12}^{(k)}$ **1 Propertify Propertify Propertify Propertify Propertify**

the damage that occurs in the concrete layers and is given al parameters obtained from the material by Equations (4) – (8) . When the layer is steel, the stress– strain relationship incorporates plastic strain after the behav-
yield limit.

exhibits linearly elastic-perfectly volving materials subject to damage and plasticity. elastic only for laminates formed by orthotopic or isotropic maextension of the classical theory to address problems interials [16]. However, this theory can be considered as an

2.2 Internal Stress Resultants in the Laminated Section

before yield deformation is assumed to be ߪ = ߝ, se ߝ > ݕߝ, where is the elastic modulus of the steel. After yielding ߝ ≤ ݕߝ , the stress is assumed to be constant, ߪ = ݕ , where ݕ is the the steel. After yielding ߝ ≤ ݕߝ , the stress is assumed to be constant, ߪ = ݕ , where ݕ is the yielding stress of the steel and exhibitslinearly elastic–perfectly plasticbehavior. er–Bernoulli Beam theory in a single stress component single stress component ݔߪ in Equation (9) used for the analysis of beams, is believed to have The model used in the study, which appears in the Euler–Bernoulli Beam theory in a $\left|\frac{h}{2}\right|$ $\left|\frac{\text{layer 2}}{2}\right|$ $\left|\frac{1}{z_2}\right|$ $\left|z_1\right|$ in Equation (9) used for the analysis of beams, is believed $\frac{1}{2}$ The model used in the study, which appears in the Eul- $\left| \frac{2}{2} \right|$ layer 3 $\left| \frac{z_2}{z_3} \right|^{2}$ in Equation (5) used for the dialityshs of beams, is believed λ below. α and α . $\mathcal{X} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ is the analysis of the analysis of the analysis of below \mathbb{Z}_2 \mathbb{Z}_1 to have only one non-zero strain component \mathcal{E}_u as given single stress component ݔߪ in Equation (9) used for the analysis of beams, is believed to have

$$
\sigma_x = Q_{11}^{(k)} \varepsilon_x \tag{13}
$$

 $\overline{}$ The internal stresses in the generic cross-section of the The internal stresses in the generic cross-section of the $\overline{}$ The internal stresses in the generic cross-section of the analysis of the analysis of believed to have $\overline{}$

$$
h_{k} = Z_{k+1} - Z_{k}
$$

$$
N = \int_{A} \sigma_{x} dA = b \int_{-h/2}^{h/2} \sigma_{x} dz
$$
 (14)

$$
M = \int_A \sigma_x z dA = b \int_{-h/2}^{h/2} \sigma_x z dA
$$
 (15)
Figure 1. Discretization of the laminated beam where *N* denotes the normal force: *M* denotes the bending

Figure 1. Discretization of the laminated beam where N denotes the normal force; M denotes the bending moment: 1 $\frac{1}{2}$ this its height. $\cdot, \, \cdot$ where *N* denotes the hornar force, *M* denotes the octunity
moment; *b* denotes the width of the section; and *h* denotes
its height scretization of the laminated beam
where N denotes the normal force; M denotes the bending
ate theory establishes that the laminae
moment: b denotes the width of the section; and b denotes $\frac{1}{2}$ moment: *b* denotes the κ and κ denotes

cess, different materials are considered. With the develop-
ment of the addition, we obtain Equations (16) and (17), $\begin{bmatrix} 0 \\ \epsilon_x \\ 0 \end{bmatrix}$ ($\begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix}$) and (a) which are compact expressions of the results expression of the section o by adding the contribution from each familia, in this process, different materials are considered. With the developns in each layer of the by adding the contribution from each lamina; in this pro- $\left\{\begin{matrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_y \end{matrix}\right\}$ (9) which are compact expressions of the resultants. isotropic materials, the following rela-
evaluation of these internal stress resultants is performed ed on the generalized Hooke's law for

For a laminated cross section, as shown in Figure 1, the

isotropic meterials, the following relation of the state of materials, the following rela-
evaluation of these internal stress resultants is performed ϵ_{x}) ment of the addition, we obtain Equations (16) and (17), trains in each layer of the by adding the contribution from each lamina: in this pro- F_{SUSY} and F_{SUSY} in Figure 1, the section of ment of the addition, we obtain Equations (16) and (17),

$$
N = A_{11} \frac{\partial u_0}{\partial x} - B_{11} \frac{\partial^2 w_0}{\partial x^2}
$$
 (16)

$$
M = B_{11} \frac{\partial u_0}{\partial x} - D_{11} \frac{\partial^2 w_0}{\partial x^2}
$$
 (17)

where where $\frac{1}{2}$ where where \overline{a}

(10)
$$
A_{11} = b \sum_{k=1}^{NC} Q_{11}^{(k)} (z_{k+1} - z_k)
$$

(11)
$$
B_{11} = \frac{b}{2} \sum_{k=1}^{NC} Q_{11}^{(k)} (z_{k+1}^2 - z_k^2)
$$

(19)

(11)
$$
B_{11} = \frac{b}{2} \sum_{k=1}^{NC} Q_{11}^{(k)} (z_{k+1}^2 - z_k^2)
$$
 (19)

$$
D_{11} = \frac{b}{3} \sum_{k=1}^{NC} Q_{11}^{(k)} (z_{k+1}^3 - z_k^3)
$$
 (20)

present in quantities $Q_{11}^{(k)}$, $Q_{12}^{(k)}$, $Q_{22}^{(k)}$ and $Q_{66}^{(k)}$ incorporates width of section b of the beam, which is not present in the minate; and v is Poisson's ratio. they are referred to in the classical laminate theory ^[16]. $\frac{1}{2}$ incorporates width of section *b* of the beam, which is not present in the expectively, which bending stiffness [B], and coupling [D], respectively, as use Deigneer's ratio $\log D_{11}$ are associa loosed here, the elastic modulus However, note that Equations (18)–(20) incorporate the $\begin{bmatrix} \text{(a)} \\ \text{(b)} \end{bmatrix}$ (12) The quantities A_{11} , B_{11} , and D_{11} are associated with the the longitudinal and transverse term $[1, 1]$ of the matrices of extensional stiffness $[A]$, **2.3 Principle ongitudinal** and transverse **box term** [1, 1] of the terial, respectively, which bending stiffness [B], and coupling [D], respectively, as
to and wis Beisson's ratio $\mathcal{G}^{(k)}$ incorporates in the structure of the total virtual work of the total virtual work of the total work $Q_{22}^{(k)}$ and $Q_{66}^{(k)}$ incorporates width of section b of the beam, which is not present in the material, respectively, which bending stiffness [B], and coupling [D], respectively, as $\sum_{i=1}^{N}$ The quantities A_{11} , B_{11} , and D_{11} are associated with the μ _{incorporation}, which belong sumess $\mathbf{[b]}$, and coupling $\mathbf{[b]}$, respectively, as [B] and coupling [D] respectively as erial, respectively, which bending stiffness [B], and coupling [D], respectively, as \sim card via Poisson's ratio \mathcal{L} study, FEM was used to model the principle of virtual the prin

equations of classical laminate theory. *p* and *p* and *p* and *p* and *q* and *q* represent to the model: integration by parts. transverse animale theory.

2.3 Principle of Virtual Works $u_0 = \bar{u}_0$ and $\delta u_0 =$

In this study, FEM was used to model the laminated
 $w_0 = w_0$ and $\omega w_0 = \omega w_0$ beam, and the principle of virtual work was used to write equilibrium equations and transform the continuous prob- $\frac{a}{1}$ lem into a discrete problem. w nere Q

Given that the structural system will be in equilibrium, if the total virtual work of the applied forces is zero, for any compatible virtual (and infinitesimal) displacement, $\frac{\partial x}{\partial x}$ $\frac{\partial x}{\partial x}$ the initial problem is determining the virtual work done by \overrightarrow{a} Discretization by f the internal forces and the virtual work done by external 2.4 Discretization forces. μ tem will be in equilibrium. where *Q* is the shear force in the section obtained from the equilibrium of the differential element $2.4.1$ Dete

The virtual work done by the internal forces for the \overline{H} problem is given by: \mathbf{v} internatively for the

$$
\delta W_{int} = \int_{V} \sigma_{x} \delta \varepsilon_{x} dV \tag{21}
$$

where $\delta \varepsilon_x$ is the variation in the strain component ε_x ; and nodes at its end V is the volume of the beam. V is the volume of the beam. $V = V \cdot V \cdot V = V \cdot V \cdot V$ is the shown in Figure 1, we obtain the following.

the following. $g.$ 0ݓ2[∂] V is the volume of the beam.
For the laminated section shown in Figure 1, we obtain used the following. **2.4.1 Determination of the Stiffness Matrix**

$$
\delta W_{int} = \int_0^L \left[\left(A_{11} \frac{\partial u_0}{\partial x} - B_{11} \frac{\partial^2 w_0}{\partial x^2} \right) \delta \left(\frac{\partial u_0}{\partial x} \right) - \left(B_{11} \frac{\partial u_0}{\partial x} \right) \right] dx
$$
 and linear poly
$$
- D_{11} \frac{\partial^2 w_0}{\partial x^2} \delta \left(\frac{\partial^2 w_0}{\partial x^2} \right) dx
$$
 (22) and linear polynomials (22)

The FEM applicat
The virtual work done by the external forces, assuming
librium equations of mat the foads are applied three ty to the data of the structure ture to produce bending, is given by:

Where *IK* and *M* and *M* denotes the structure; *M* denotes the structure; *M* is the global stiffness matrix of the structure. it the loads are applied directly to the axis of the structhat the loads are applied directly to the axis of the struc-
 $[K](D) = \{F\}$ (3) ture to produce bending, is given by:
 $[K]_1U_3 = \{F\}$ (30)

where [K] is the global stiffness matrix of the structure, a The FEM application
The virtual work done by the external forces, assuming
librium equations of ty that ture to produce bending, is given by:
 $[K]\{D\} = \{F\}$

where $[K]$ is the global s [∂]^ݔ െ ¹¹

$$
\delta W_{ext} = \int_0^L \left[p(x) \delta u_0 + q(x) \delta w_0 \right] dx
$$
\nwhere [h] is the function of the function of the
\n
$$
+ \left[\bar{F}_x \delta u_0 + \bar{F}_z \delta w_0 - \bar{M} \delta \left(\frac{\partial w_0}{\partial x} \right) \right]_0^L
$$
\n(23) channel proped as a
\nthe global vector

domain according to the axial x axis and transverse z axis, all displacements, obtained beam ends $(x = 0$ and $x = L)$, respectively; and \bar{M} repre-
sents the systemal moments equiled at the sense and $[K] = \sum_{k=0}^{N_E} [k_{el}]$ ectively, \mathbf{r}_x and \mathbf{r}_z represent the forces applied at the tions (30). s the external moments applied at the same ends. the global vect
where $p(x)$ and $q(x)$ represent the distributed loads of the
loads acting or al displacements, obtained by solving the system of Equa-
respectively; \vec{F}_x and \vec{F}_z represent the forces applied at the tions (30). $\frac{1}{1}$ ends $\frac{1}{1}$ where $p(x)$ and $q(x)$ represent the distributed loads of the global
domain according to the avial y ovie and transverse a quie sents the external moments applied at the same ends. $[K] = \sum_{1}^{N_E} [k_{el}]$ ^ݔ and lj $\frac{1}{\sqrt{1-\frac{1$ $|K| =$ ⁰ ൌ ݐܹ݅݊ߜ to the axis of the structure to the structure to the structure beam ends $(x = 0 \text{ and } x = L)$, respectively: and M repre- S , $\mathbf{r}_1 - \mathbf{r}_2$

 $\mathbf I$ By applying the equilibrium condition imposed by the where NE is the number By applying the equinorium condition imposed by the variance principle of virtual work (PTV), that is, $\delta W_{int} = \delta W_{ext}$, the the model: \mathbf{r} previously making the variations of displacements in the $\left[\begin{array}{cc} \frac{A_{11}}{I_1} & 0 \end{array}\right]$ gration by parts. $\left| \begin{array}{cc} 0 & \frac{12D_{11}}{L_{\text{eff}}^3} \end{array} \right|$ (27) , 272.2 , 272.2 by applying the equinorium condition imposed by the where *NE* is the number
principle of virtual work (PTV), that is, $\delta W_{int} = \delta W_{ext}$, beam discretization; and the differential equations of the problem can be instituted, beam $\frac{1}{100}$ $\frac{1$ $\frac{1}{\sqrt{2}}$ parts and $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}}$ domain portions in Equations (22) and (23) through inte-
 $\begin{bmatrix} \frac{a_{11}}{L_{el}} & 0 & -\frac{b_{11}}{L_{el}} & -\frac{a_{11}}{L_{el}} & 0 & \frac{b_{11}}{L_{el}} \\ 0 & 12D_{11} & 6D_{11} & 0 & 12D_{11} & 6D_{11} \\ 0 & 0 & 12D_{11} & 6D_{11} & 0 & 12D_{11} \end{bmatrix}$ ي البراء الماضية.
البراء البراء ال principle of virtual work $(1 \times t)$, that is, $\sigma v_{int} = \sigma v_{ext}$, begin discretization, and integration by parts. by applying the equinorium condition imposed by the where *NE* is the humor
principle of virtual work (PTV) that is $\delta W_{int} = \delta W_{ext}$ beam discretization; and principle of virtual work (PTV), that is, $\delta W_{int} = \delta W_{ext}$, beam discretization; and integration by parts. previously making the variations of displacements in the principle of virtual work (PTV), that is, $\delta W_{int} = \delta W_{ext}$, beam discretization $\frac{1}{2}$ and direct matrix equations of the problem can be instituted, begin element, which is given by previously making the variations of displacements in the $\frac{1}{2}$ domain portions in Equations (22) and (23) through interestion by parts gration by parts. $\overline{}$

This result in the system of differential equations asso-
existed with the model: اند **2.4.1 Determination of the Stiffness Matrix** . د د دي.
ا $\frac{dA}{dt}$ is the shear force in the shear from the section of the differential elements of the different ciated with the model: v_{H} or v_{H} or v_{H} or v_{H} or v_{H} or v_{H} \mathbf{e} l: .
البراهيم المركز ال
المركز المركز الم T value with the model. uations asso-

$$
A_{11} \frac{\partial^2 u_0}{\partial x^2} - B_{11} \frac{\partial^3 w_0}{\partial x^3} = -p(x) \quad \text{or} \quad \frac{\partial N}{\partial x} = -p(x) \quad (24)
$$

$$
B_{11} \frac{\partial^3 u_0}{\partial x^3} - D_{11} \frac{\partial^4 w_0}{\partial x^4} = q(x) \qquad \text{or} \qquad \frac{\partial^2 M}{\partial x^2} = q(x) \qquad (25) \qquad \begin{array}{ccc} & & L_{el} & & L_{el} \\ \frac{B_{11}}{L_{el}} & \frac{6D_{11}}{L_{el}^2} \end{array}
$$

Because of the application of PTV, the following Du boundary conditions are extracted at $x = 0$ and $x = L$, in-
element, the r Because of the application of PTV, the following During the process ݔ߲ boundary conditions are extracted at $x = 0$ and $x = L$, in-

pry. herent to the model: B_1 applied the following boundary condition of P $\text{if } \log y.$ herent to the model: making the variations of displacements in the domain portions in Equations (22) and (23) through herent to the model: repent to the model.

that is, \overline{a} , the differential equations of the problem can be instituted, problem can be instituted, previously \overline{a}

⁰ ^ݔ[∂]

$$
u_0 = \bar{u}_0 \text{ and } \delta u_0 = \delta \bar{u}_0 \qquad \text{or} \qquad N = \bar{F}_x \quad (26)
$$

 $\mathcal{L}_{\mathcal{A}}$ applying the equilibrium condition imposed by the principle of virtual work (PTV), $\mathcal{L}_{\mathcal{A}}$

integration by parts.

 $\mathcal{B}(\mathcal{A})$ application of $\mathcal{B}(\mathcal{A})$ the following boundary conditions are extracted at $\mathcal{B}(\mathcal{A})$

$$
w_0 = \bar{w}_0 \text{ and } \delta w_0 = \delta \bar{w}_0 \qquad \text{or} \qquad Q = \bar{F}_z \tag{27}
$$

FeM was used to model the laminated

$$
\frac{\partial w_0}{\partial x} = \frac{\partial w_0}{\partial x} \text{ and } \delta \left(\frac{\partial w_0}{\partial x} \right) = \delta \left(\frac{\partial w_0}{\partial x} \right) \text{ or } M = \bar{M} \tag{28}
$$
\nwhere *O* is the shear force in the section obtained from the

storm the continuous prob-
where Q is the shear force in the section obtained from the ium, equilibrium of the differential element of the beam. where χ is the shear force in the section obtained from the \det of the hoom

$$
Q = \frac{\partial M}{\partial x} = B_{11} \frac{\partial^2 u_0}{\partial x^2} - D_{11} \frac{\partial^3 w_0}{\partial x^3}
$$
 (29)

zork done by external **2.4 Discretization by the Finite Element Method** where \mathbf{a} is the section obtained from the section of the differential element of the differential elements of the differential elements of the differential elements of the differential elements of the differential el ation by by the Fire **Finite Element Method**

2.4.1 Determination of the Stiffness Matrix
rnal forces for the $\mathcal{L}(\mathcal{L})$. $\mathcal{L}(\mathcal{L})$. $\mathcal{L}(\mathcal{L})$. $\mathcal{L}(\mathcal{L})$ 2.4.1 Determination of the Stiffness Matrix

 $V \propto \sigma_x \delta \epsilon_x dV$ (21) previous subsections ^[17]. This element is delimited by two section shown in Figure 1, we obtain used to represent the displacements along the finite elewing.
wing.
 $\lim_{u \to 0} \left(\frac{u}{v} \right)^2$ (i) $\lim_{u \to 0} \left(\frac{u}{v} \right)^2$ (i) $\lim_{u \to 0} \left(\frac{u}{v} \right)^2$ $\frac{\partial w_0}{\partial x}$ was obtained by deriving from $w_0(x)$. given by:

Herein, the classical beam element was chosen for

treatment using FEM for formulation developed in the
 $\sigma_n \delta \epsilon_n dV$ these nodes: u_0 , w_0 , and $\frac{\partial u}{\partial x}$. The interpolation function $\delta W_{int} = \int_0^L \left[\left(A_{11} \frac{\partial u_0}{\partial x} - B_{11} \frac{\partial^2 w_0}{\partial x^2} \right) \delta \left(\frac{\partial u_0}{\partial x} \right) - \left(B_{11} \frac{\partial u_0}{\partial x} \right) \right]$ and linear polynomials for $u_0 = u_0(x)$. Finally, the rotation $-\frac{B_{11}}{\partial x^2} \left(\frac{\partial}{\partial x} \right) - \left(B_{11} \frac{\partial}{\partial x} \right)$ (22) and the exponential polynomials for $u_0 = u_0(x)$. Finally, the foldom Herein, the classical beam element was chosen for e strain component ε_x ; and nodes at its ends, with three degrees of freedom at each of . This element is defined by two nodes at its ends, with elements at its ends, with α these nodes: u_0 , w_0 , and $\frac{\partial w_0}{\partial x}$. The interpolation functions $\mathcal{L}_{\mathcal{A}}$ Herein, the classical beam element was chosen for nodes at its ends, with three degrees of freedom at each of two chosen for the chosen for the chosen for the chosen for \mathcal{L} $n \geq 0$

 $\frac{1}{2}$
The FEM application generates a system of nodal equi- $\frac{1}{2}$ are $\frac{1}{2}$ and $\frac{1}{2}$ $\frac{1}{2}$ and $\frac{1}{2}$ $\frac{1}{2}$ and $\frac{1}{2}$ are external forces, assuming librium equations of type

$$
[K]\{D\} = \{F\} \tag{30}
$$

 $\delta W_{ext} = \int_0^L [p(x)\delta u_0 + q(x)\delta w_0] dx$

function of both the geometry of the beam and the me- $+\left[\bar{F}_x \delta u_0 + \bar{F}_z \delta w_0 - \bar{M} \delta \left(\frac{\partial w_0}{\partial x}\right)\right]_0$ assembly of elements, as shown in Equation (31); $\{F\}$ $\left(\begin{array}{cc} \frac{\partial x}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial y}{\partial z} & \frac{\partial z}{\partial x} \end{array}\right)$ the global vector of loads, containing the equivalent nodal the axial x axis and transverse z axis,
al displacements, obtained by solving the system of Equa-
 \vec{F}_z represent the forces applied at the For a languarity of the evaluation of (23) and the evaluation of (23) and (23) and (23) and (23) and $($ r^{25} channel properties of the materials, and is given by a contribution from each land $\left(25\right)$ contribution from each land $\left(21\right)$ (m) $+\left[\bar{F}_x \delta u_0 + \bar{F}_z \delta w_0 - \bar{M} \delta \left(\frac{\partial w_0}{\partial x}\right)\right]_0^L$ assembly of elements, as shown in Equation (31); $\{F\}$ is (23) chanical properties of the materials, and is given by:
(23) $\log \log \left(\frac{\text{odd}}{\text{odd}} \right)$ are the structure: and Ω is the vector the $\frac{1}{100}$ tions (30). $\frac{1}{\sin(30)}$ the distributed loads of the loads acting on the structure; and $\{D\}$ is the vector of nod-
ly axis and transverse z axis $\frac{\log(x)}{\log(x)}$ and $\frac{q(x)}{\log(x)}$ represent the distributed loads of the loads acting on the structure; and $\{D\}$ is the vector of nod-
according to the axial x axis and transverse z axis, (23) chanical properties of the materials, and is given by the the global vector of loads, containing the equivalent nodal
t the distributed loads of the $\frac{1}{2}$ assembly of elements, as shown in Equation (31); $\{F\}$ is $\frac{1}{2}$ axis and uansverse $\frac{2}{2}$ axis, all displacements, obtained by solving the system of Equa-
esent the forces applied at the $\frac{1}{2}$ i.e.s. (20) x_{max} at displacements, obtained by solving the system of Equa- \bar{M} representing the materials of the assembly of the assembly of \bar{M} represents, assembly of \bar{M} shown in Equation (31); ܨ is the global vector of loads, containing the equivalent nodal loads

$$
[K] = \sum_{i=1}^{N} [k_{el}]
$$
 (31)

of the same ends.

um condition imposed by the where *NE* is the number of finite elements defined in the

EVO details $\frac{SW}{d} = \frac{SW}{d}$ blem can be instituted, beam element, which is given by quinorially condition imposed by the where γE is the hallfoed of finite elements defined in the problem can be instituted. Problem can be instituted in the problem can be instituted. Problem can be interesting to the p work (PTV), that is, $\delta W_{int} = \delta W_{ext}$, beam discretization; and $[k_{el}]$ is the stiffness matrix of the $\frac{1}{2}$ is the stationary of the stationary and $\frac{1}{2}$ is the stations matter of the am discret ݈݁ܮ on; and $[k_{el}]$ is the the surfliess matrix of 1 where *NE* is the number of finite elements defined in the beam discretization; and $[k_{el}]$ is the stiffness matrix of the

making the variations of displacements in the
ptrions in Equations (22) and (23) through inte-
parts.

\nwith the system of differential equations asso-
the model:

\n
$$
B_{11} \frac{\partial^3 w_0}{\partial x^3} = -p(x)
$$
\nor

\n
$$
\frac{\partial N}{\partial x} = -p(x)
$$
\nor

\n
$$
\frac{\partial N}{\partial x} = q(x)
$$
\nor

\n
$$
\frac{\partial N}{\partial x} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^2} = q(x)
$$
\nor

\n
$$
\frac{\partial^2 M}{\partial x^
$$

TV, the following During the process of applying loads on the structural $= L$, in- element, the matrix $[k_{el}]$ can be different even for ele p_{max} is the following solution, for an initial (predicted) solution, $\frac{p_{\text{max}}}{p_{\text{max}}}$ ted at $x = 0$ and $x = L$, in-
element, the matrix $[k_{el}]$ can be different even for eleacted at $x = 0$ and $x = L$, in-
element, the matrix $[k_{el}]$ can be different even for ele-݈݁ܮ 611 applying loa PTV, the following During the process of applying loads on the structural acted at $x = 0$ and $x = L$, in-
element, the matrix $[k_{el}]$ can be different even for elements of the same length *Lel* because the cracking process of the concrete or the yielding of the reinforcement causes the damage variable to assume different values along the length or height of the beam.

2.4.2 Nonlinear Analysis

To reach the final equilibrium solution, incremental application of external loads was performed to obtain an initial (predicted) solution, followed by an iterative Newton–Raphson using force or displacement control process. Table 1 summarizes the flowchart of the program.

2.5 Model Validation

The experimental results obtained by Álvares $^{[15]}$ were an used to validate the proposed model. Reinforced concrete beams with different reinforcement ratios were experimentally investigated to evaluate their failure form when subjected to a four-point bending test. The experimental test of the beams was performed with load control such that the test was interrupted when the breaking load was reached.

The beams evaluated by Álvares^[15] had a rectangular section measuring 120 mm \times 300 mm, with a span of the lengt 2400 mm and loads located 800 mm from the support, as shown in Figure 2a and 2b. and the support, as in an above the support, as in shown in Figure 2a and 20. $\mathbf{1}$ β 011, α

The reinforcement rate of the beams was varied such The reinfereement rate of the beams was varied such ure (under-reinforced section), flexural compression an in Which right 2a and 20.

failure (over-reinforced section), and simultaneous failure (optimized section), could be evaluated. The upper reinforcement of all beams consisted of two bars with a diameter of 5 mm. The lower reinforcement varied based on the type of failure expected for the beam: i) for the under-reinforced section beam (Figure 2b), three bars with a diameter of 10 mm were used $(A_s = 236$ mm²); ii) for the optimized section beam (Figure 2c), five bars with a diameter of 10 mm were used $(A_s = 393 \text{ mm}^2)$; iii) for the over-reinforced section beam (Figure 2d), seven bars with $\frac{300}{2}$ a diameter of 10 mm were used $(A_s = 550 \text{ mm}^2)$. For beam reinforcement, the following properties were assumed for steel: $E_s = 196 \text{ GPa}, f_y = 500 \text{ MPa}, \text{ and } f_u = 500 \text{ MPa}.$ $\frac{1}{2}$ (over-remoted section), and simulations failsteel: $E_s = 196 \text{ GPa}, f_y = 500 \text{ MPa}, \text{ and } f_u = 500 \text{ MPa}.$

reinforcement varied based on the type of failure expected for the beam: i) for the under-

For concrete modeling, an elastic modulus of 29.2 GPa and the following parameters necessary for the Mazar damage model, defined by Álvares $\begin{bmatrix} 1.5 \end{bmatrix}$, were used. $A_t =$ were experi-
995, $B_t = 8000$, $A_c = 0.85$, $B_c = 1620$ and $\varepsilon_{d0} = 0.00007$. Ade notice model, defined by Álvares $^{[15]}$, were used. $A_t =$ re form when ditionally, a Poisson's ratio of 0.2 was assumed.

attionary, a roisson's ratio of 0.2 was assumed.
A convergence study (verification process) of the disonvergence study (vermed ton process) or the disload control such cretization parameters of the load-displacement solution breaking load was was performed to determine the finite element mesh of the beams by varying the number of layers (10, 20, and the beams by varying the number of layers (10, 20, and $r_{\rm eff}$ the simulation of the three beams were a discretization of ϵ 40 layers) of the cross section, the number of elements for initial load step $(0.5, 1.0, \text{ and } 2.0 \text{ kN})$, and the tolerance of the length of the beams $(12, 24, 36, \text{ and } 48 \text{ elements})$, the the iterative process $(10^{-3}, 10^{-4}, \text{ and } 10^{-6})$. Therefore, the ed such investigation recommended for the simulation of the three If tension fail-
learns were a discretization of 20 layers and 36 elements, compression an initial load step of 1 kN and tolerance of 10^{-6} . had a rectangular 40 layers) of the cross section, the number of elements for \overline{a} cretization parameters of the load-di \mathbf{r} . eams by v ݈݁ܮ $\ddot{}$ ng the number of by varying the number of layers (10, 20, and the length of the beams $(12, 24, 36, \text{ and } 48 \text{ elements})$, the terative pro process $(10^{-3}, 10^{-4}, \text{ and } \frac{1}{2})$ s $(10^{-3}, 10^{-4}, \text{ and } 10^{-6})$. Therefore, the -displacement solution ݈݁ܮ $\ddot{}$ b, 1.0, and 2.0 kN), and the tolerance of

Pre-cracking phase $(D = 0)$	Post-cracking phase $(D > 0)$
	1. Calculation of the stiffness matrix in relation to the last equilibrium
	configuration using displacements, strains, stresses, and the updated damage
	variable;
	2. Update load from load increment $\{AF\}$;
	3. Solve the system of Equation (30);
1. Calculation of the stiffness matrix;	4. Check convergence through the unbalanced force (external forces minus)
2. The load is updated from the load increment $\langle AF \rangle$;	internal forces);
3. Solve the system of Equation (30);	5. If there is no convergence, the stiffness of the structure must be updated
4. Return to step 1;	and then the increment of the nodal displacements $\{AD\}$ must be calculated
	through the unbalance force. Subsequently, the displacements $({\Delta}D) + {D}$ and
	the damage variable are updated. Finally, the unbalanced force is updated and
	the convergence is verified (external force minus internal force). Repeat this
	process until the solution converges to that charge level.
	6. After convergence, go back to step 1.

Table 1. Iterative process of obtaining a solution

Figure 2. Experimental setup ^[15]: a) four-point bending test; b) under-reinforced beam; c) optimized beam; d) over-reinforced beam.

3. Results and Discussion

For the three types of beams, Figures 3~5 show the force–displacement curves obtained experimentally by Álvares [15] and the respective numerical results obtained from the proposed model. The experimentally acquired force–displacement curves exhibited the typical behavior of reinforced concrete beams subjected to bending failure, along with the identification of three stages (Figure 3). In Stage I, the concrete was undamaged, and the stiffness of the EI_I beams was because of the combined action of concrete and steel. The cracking of concrete indicates the end of this stage. The cracking load is defined by the tensile strength of the concrete.

In stage II, the curve initially exhibits nonlinear behavior that is characterized by the appearance of multiple cracks on the lower face of the beam. Gradually, stress is transferred to the steel bars, which provide the tensile strength of the beam. As the load increases, a second linear section is formed whose slope represents the stiffness EI_{II} of the cracked beam and is defined primarily by the reinforcement rate. However, the cracked concrete can contribute to the stiffness in a phenomenon called the tension-stiffening effect^[18].

Stage III begins with a further reduction in the stiffness and a trend to stabilize the force until the beam fails. The reinforcement rate of the beam affects the force and displacements that define the beginning and end of stage III, which can lead to three types of failure associated with deformations in steel and concrete at the instant of beam collapse.

For stages I and II, a good approximation between the experimental load–displacement curves and the curves obtained using the proposed model, wherein the damage model is associated with the classical theory of laminates, can be confirmed by comparing the numerical results with the experimental results. However, the experimental curve presents an ultimate displacement during the beam test that is smaller than that predicted by the numerical result. This is because the load control used in the experiment halts the test when the maximum load is reached.

In the numerical model, taking the limits of deformation presented in Figure 6 as a reference, the beam failure was established by monitoring the strains in the most compressed concrete layer and in the most stressed steel layer. The ultimate limit states of a reinforced concrete beam can be established when the strain in the concrete reaches a value ε_{cu} = 0.35% because of compression failure, and/ or by tensile failure when the strain in the steel reaches a value ε_{su} = 1.00% caused by crushing the compressed section. Balanced beams fail because of crushing of the compressed region; however, the strain in the steel is equal to or less than the yield strain ε_{sy} . When the beam cross-section and reinforcement ratio are optimally designed, failure occurs simultaneously in the top compressed layer and the most tensioned reinforcement section.

The proposed numerical model allows for the monitoring of the strains of the materials of the beam and the identification of the failure mechanism, as shown in Figure 7. In the over-reinforced beam, failure occurs by crushing the compressed region. This beam has the highest failure load, of the order of 73 kN, but a lower total displacement than the other beams analyzed. For the under-reinforced beam, the maximum load obtained was 81% lower than the load observed for the over-reinforced beam, and the deformation was 1.2 times greater. The optimized beam presents a load 46% less than the load observed for the over-reinforced beam, but with a deformation 1.3 times greater. In addition, this beam presents the best use of materials, which contributes to the reduction of energy consumption and non-renewable materials, thereby increasing the sustainability of the structures.

Figure 4. Flexural behavior of optimized beam

Figure 5. Flexural behavior of over-reinforced beam

The results obtained by the numerical model indicated a variation in the stress–strain behavior of each beam, as shown in Figure 8. For under-reinforced beams, the compressive stresses did not reach the peak stress but the stress–strain behavior was observed in the nonlinear regime at failure, thus indicating the appearance of damage to the stiffness of the concrete. As the strains in the reinforcement reached their maximum value, the neutral axis approached the upper surface of the beam section. For the other beams, the concrete had reached its ultimate strain, and the beam's neutral axis was close to the centroid of the cross-section.

Figure 6. Strain limits for steel and concrete in the beam cross section

Figure 7. Theoretical identification of limiting strains of steel and concrete

Evidently, the proposed model for the behavior of tensioned concrete considers the contribution of cracking concrete (below the neutral axis), in contrast to design codes for reinforced concrete structures, even though the tensile stress value is low when comparing the stresses in the reinforcement and even in the compressed concrete.

Figure 8. Stress–strain diagrams of concrete at failure of beam.

The variation in the stress–strain behavior is a function of the evolution of the damage parameter. As established by Equation (3), when the deformations exceed the limit value, ϵ_{d0} , there is a gradual reduction in the stiffness of the beam owing to cracking of the concrete. Figure 9 shows the variation in the damage parameter $(1-D_c)$ for the three types of beams investigated, with the increase in the vertical displacement of the beam. Initially, the value of $(1-D_c)$ was equal to unity because there was no damage to the compressed concrete. With increasing displacement, a reduction in this parameter was verified; however, it was affected by the reinforcement ratio of the beam. At failure, compression damage of approximately 60% was observed for under-reinforced beams, and the damage was approximately 80% for the optimized and over-reinforced beams.

Figure 9. Variation in the damage parameter with increase in the vertical displacement of the beams.

4. Limitations of the Study

The model used in this study, within the scope of static loading, ignores shear and geometric nonlinearity effects. Furthermore, the Mazars damage model is elastic and is not appropriate for situations of cyclic loadings, which is not the case in the present study. However, the order of magnitude of the maximum transverse displacement of the beam with respect to the height is small, thus justifying the geometric linear analysis. The failure modes of the beams did not include shear failures.

5. Conclusions

The proposed model combines the classic theory of laminates and the Mazars damage model. By using FEM, it was able to evaluate the flexural behavior of reinforced concrete beams up to the failure of these elements for different rates of flexural using numerical simulation of the reinforced concrete beams under four-point bend tests. This was possible because the strategy of incorporating in the finite elements, the lamination of the transverse section, and the physical nonlinearity of the materials by continuous damage mechanics allowed the following of the stress and strain state of each layer of material, whether concrete (with its progressive cracking) or the reinforcement (even in the yielding).

Therefore, despite the relative simplicity of the proposed model, its potential to predict the behavior of reinforced concrete beams under bending was demonstrated, thereby allowing a precise identification of deformations and rupture criteria. The numerical model allowed the identification of the failure form of each type of reinforced concrete beam analyzed, through the prediction of the neutral line variation and the determination of the stress-strain behavior. In this way, the model can be used to predict the behavior of structural elements subjected to bending and lead to optimized designs, with greater safety and lower cost.

Author Contributions

JMFL: methodology, numerical modelling, formal analysis, writing - original draft; PRLL: conceptualization, investigation, data curation, formal analysis, writing review & editing; GJBS: numerical modelling, formal analysis, writing - original draft.

Conflict of Interest

No conflict of interest.

Funding

This research was funded by CNPq, grant numbers 313693/2019-6 and 408135/2021-2, and State University of Feira de Santana, grant numbers 034/2021 and 064/2021.

References

- [1] Wang, T., Hsu, T., 2001. Nonlinear finite element analysis of concrete structures using new constitutive models. Computer& Structures*.* 79, 2781-2791.
- [2] Assan, A., 2002. Nonlinear analysis of reinforced concrete cylindrical shells. Computer&Structures*.* 80, 2177-2184.
- [3] Tao, X., Phillips, D., 2005. A simplified isotropic damage model for concrete under bi-axial stress states. Cement & Concrete Composites*.* 27, 716-726.
- [4] Mazars, J., Kotronis, P., Ragueneau, F., et al., 2006. Using multifiber beams to account for shear and torsion. Applications to concrete structural elements. Computer Methods in Applied Mechanics and Engineering*.* 195, 7264-7281.
- [5] Butean, C., Heghes, B., 2020. Flexure Behavior of a two layer reinforced concrete beam. Procedia Manufacturing. 46, 110-115.
- [6] Liu, C., Yang, Y., Wang, J., et al., 2020. Biaxial reinforced concrete constitutive models for implicit and explicit solvers with reduced mesh sensitivity. Engineering Structures. 219, 110880.
- [7] Tjitradi, D., Eliatun, E., Taufik, S., 2017. 3D ANSYS numerical modeling of reinforced concrete beam behavior under different collapsed mechanisms. International Journal of Mechanics and Applications. 7(1), 14-23.
- [8] Gorgogianni, A., Elias, J., Le, J.L., 2020. Mechanism-based energy regularization in computational modeling of quasibrittle fracture. Journal of Applied Mechanics. 87(9), 091003.
- [9] Arruda, M.R.T., Pacheco, J., Castro, L.M.S., et al.,

2022. A modified mazars damage model with energy regularization. Engineering Fracture Mechanics. 259, 108129.

- [10] Carrera, E., Augello, R., Pagani, A., et al., 2021. Component-wise approach to reinforced concrete structures. Mechanics of Advanced Materials and Structures. 1-19.
- [11] Arruda, M.R.T., Castro, L.M.S., 2021. Non-linear dynamic analysis of reinforced concrete structures with hybrid mixed stress finite elements. Advances Engineering Software*.* 153, 102965.
- [12] Leone, F.A., Justusson, B.P., 2020. Effects of characteristic element length on fracture energy dissipation in continuum damage mechanics models. Journal of Composites Materials. 55(24), 3551-3566.
- [13] Di Prisco, M., Mazars, J., 1996. Crush-crack': a non-local damage model for concrete. Mechanics of Cohesive-frictional Materials. An International Journal on Experiments, Modelling and Computation of Materials and Structures. 1(4), 321-347. DOI: [https://doi.org/10.1002/\(SICI\)1099-1484](https://doi.org/10.1002/(SICI)1099-1484(199610)1:4<321::AID-CFM17>3.0.CO;2-2) [\(199610\)1:4<321::AID-CFM17>3.0.CO;2-2](https://doi.org/10.1002/(SICI)1099-1484(199610)1:4<321::AID-CFM17>3.0.CO;2-2)
- [14] Mazars, J., Lemaitre, J., 1985. Application of Continuous Damage Mechanics to Strain and Fracture Behavior of Concrete. Shah, S.P. (eds) Application of Fracture Mechanics to Cementitious Composites. NATO ASI Series, vol 94. Springer, Dordrecht. DOI: https://doi.org/10.1007/978-94-009-5121-1_17
- [15] Alva, G.M.S., El Debs, A.L.H.C., Kaminski Jr, J., 2010. Nonlinear analysis of reinforced concrete structures in design procedures: application of lumped dissipation models. Revista IBRACON de Estruturas e Materiais. 3, 149-178. DOI: https://doi.org/10.1590/S1983-41952010000200003
- [16] Reddy, J.N., 2004. Mechanics of laminated composite plates and shells: theory and analysis. CRC Press. USA. pp. 831.
- [17] Cook, R.D., Malkus, D.S., Plesha, M.E., et al., 2002. Concepts and applications of finite element analysis. John Wiley & Sons. Inc. USA. pp. 719.
- [18] Martins, M.P., Rangel, C.S., Amario, M., et al., 2020. Modelling of tension stiffening effect in reinforced recycled concrete. Revista Ibracon de Estruturas e Materiais. 13, 1-21.