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ARTICLE

Optimization Processes of Tangible and Intangible Networks through the Laplace Problems for Regular Lattices with Multiple Obstacles along the Way

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ABSTRACT

A systematic approach is proposed to the theme of safety, reliability and global quality of complex networks (material and immaterial) by means of special mathematical tools that allow an adequate geometric characterization and study of the operation, even in the presence of multiple obstacles along the path. To that end, applying the theory of graphs to the problem under study and using a special mathematical model based on stochastic geometry, in this article we consider some regular lattices in which it is possible to schematize the elements of the network, with the fundamental cell with six, eight or $2(n+2)$ obstacles, calculating the probability of Laplace. In this way it is possible to measure the "degree of impedance" exerted by the anomalies along the network by the obstacles examined. The method can be extended to other regular and / or irregular geometric figures, whose union together constitutes the examined network, allowing to optimize the functioning of the complex system considered.

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1. The Representation of Complex Networks with Mathematical Models

Complex networks are networks whose structure is irregular, complex and which evolves over time and are used in various branches of science and technology, as in biochemistry, in the study of interactions in quantum field theory, in study of IT processes, topologies in geographical databases[1] and also on the web, in social networks such as Facebook and Linkedin and in the Google model.

The same term "complex" derives from the latin cum (together) - plexus (intertwined), "intertwined together": it highlights that a complex network system is composed of a set of parts connected together and "intertwined" in such a way that the final result (the effect produced) is different from the sum of the constituent parts. Therefore, the behavior of a complex system cannot be inferred by a simple analysis of the elements that compose it, but it is necessary a systematic examination of the interactions that are generated between them and the constraints that determine their operation must be carried out.

An appropriate methodological approach to the problem for the applied sciences can make use of particular discrete mathematical structures, called "graphs", which are represented by an *n-air* relationship on a finite set *S*, defined by the subsets of *S* with *n* elements which satisfy a property *P(1,..,n)*, which can be constructed in such way to be representative of the structure and the behavior of the main tangible and intangible networks, such as:

(1) WWW = *World Wide Web*;

(2) telematic or real communication networks (telephone networks, road, railway, port, logistics, airways, etc.);

(3) *networks describing ecological systems (food webs, etc.)*;

(4) biological networks (neural networks, genetic transcription networks, metabolic networks, protein networks, etc.);

(5) relationship networks (social networks, scientific collaboration networks, etc.);

(6) musical networks for composition (graphs based on notes and others on chords can be linked together by a relationship of duality to create the melody and harmony of a piece of music).

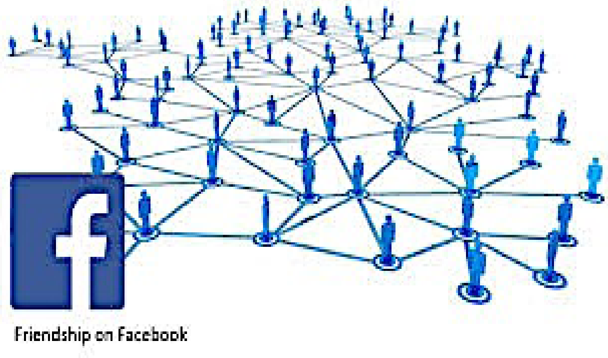

Figure 1.

From a geometric point of view, a graph is a data structure made up of a set of vertices (or nodes) which are related to each other through links, called arcs. In essence, a graph can be imagined as a set of points randomly arranged in the space R^n connected by "bridges" to the other points. So in a computer network, for example, the nodes are represented by all the devices connected to the network (such as a PC connected to the internet), while the arcs are represented by the communication channels that allow users to interact in the network. If the relationship between the elements of the represented set does not provide an order between them, there is a not oriented graph (see the following figure), while if it is foreseen, there is an oriented graph.

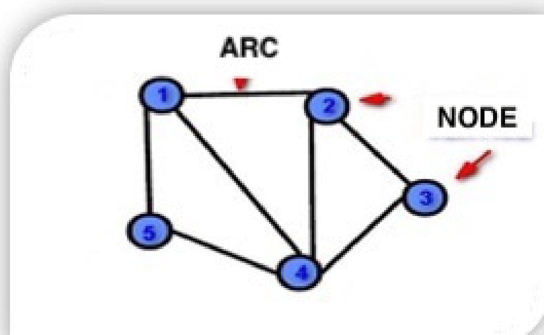

Figure 2.

An not oriented graph $G=(N,L)=G_{N,K}$ is algebraically definable through a pair of two sets *N* and *L* where the elements of are $N=\{n_1,n_2,...,n_K\}$ called nodes or vertices and the elements of $L=\{l_1,l_2,...,l_K\}$ are pairs of elements of *N*, called arcs or links. The number of elements of the set of vertices and arcs is respectively *N* and *K*. The graph is oriented if the elements of $L=\{l_1,l_2,...,l_K\}$ are ordered pairs of elements of *N*. In this case, l_{ij} indicates an arc that connects node *i* with node *j*.

Given a graph *G* with *N* nodes, the number *K* of arcs

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can vary from a minimum of 0 arcs to a maximum of *N(N-1)/2* and in this case all the nodes are adjacent (connected to each other by a arch)[1]; it is defined as connected if for each pair of distinct nodes *i* and *j*, there is a path $i \rightarrow j$, otherwise it is disconnected, for the mathematical representation a special matrix is used, called adjacency, typically indicated as *A*, of dimension *NxN* and the degree of a node *i* is calculated through the number k_i of arcs adjacent to *i*[2].

The numerical representations of a graph can be matrix or vector, the nodes of the set *N* are usually indicated with an integer. The representation of the graph most used in calculation programs is known as the outgoing star, or forward star, in which each node *i* is associated with the set of arcs it leaves, or the final nodes of these arcs, FWS (i). A similar representation of the graph called backward star (BWS) is obviously possible, which requires exactly the same amount of data. The choice between the representations depends on the algorithm for calculating the minimum paths that is adopted.

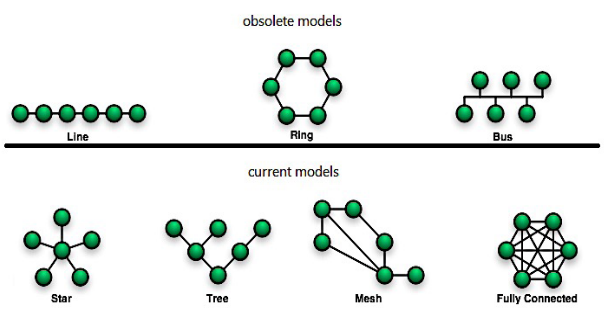

Figure 3.

A graph G(N,L) can be stored on a computer in multiple ways, the choice of which depends on the characteristics to be highlighted and the needs for use, as well as on its size and density (number of arcs compared to the number of nodes),

(1) connection matrix;

(2) incidence matrix;

(3) adjacency list.

Of these (see following diagrams), the first and the last are the most used[3] but the incidence matrix, although less effective than the other two methods from the computational point of view, is however preferable in some optimization problems due to its relative ease of representation.

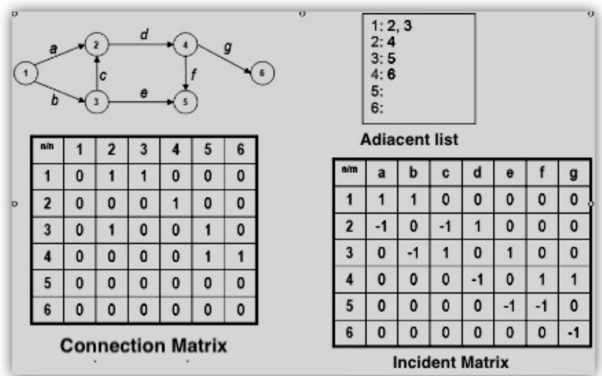

Figure 4.

The connection matrix as storage method is based on the use of a square matrix *nxn* (with *n* number of nodes in the graph), in which the generic element *(i,j)* assumes the value 1 if there is an arc that connects the node *i* with node *j*, while 0 in the other cases, so in the case of undirected graphs the matrix will be symmetric, since the element *(i,j)* is equal to the element *(j,i)*.

Furthermore, depending on the particular infrastructural network (material and / or immaterial) under examination, it is possible mathematically to "deposit" on the set of vertices or arcs (or both) some quantitative information of the arcs or vertices that represent specific "assessments" that characterize the problem and its constraints. It is possible to refer to specific models representative of the specific realities in the study, defining each time the set of variables, the objective function (or multi-objective) and the system of constraints, as in the case that we want to determine the cheapest way to transport a certain amount of an asset (for example, gas, oil, industrial or agricultural products, etc.) from one or more production nodes to one or more consumption points, through a certain transport network (hydraulic, distribution, road or railway, of the production chain, etc.)[4]. For such models of flow distribution at minimum cost, the network can be conveniently

① A graph is called complete if $K = \binom{N}{2} = N \cdot (N-1)/2$ it is denoted by K_N, while a walk G ($i \rightarrow j$) is a sequence of nodes and arcs which begin from *i* to *j* and they can cross more times.

② For example, an infrastructural network of land mobility (road or rail) can be represented through an oriented and connected graph *G=(N,L)*, in a computerized way.

③ The connection matrix is preferable in the case of very dense graphs because, compared to the adjacency list, it allows greater efficiency and immediacy of the calculations, while the latter is useful in the case of scattered graphs (i.e. the number of arcs is small compared to the number of nodes).

④ The mathematical model of the problem is here of a general nature because it lends itself to representing cases that have nothing to do with the shipment of goods and, therefore, the more abstract notion of flow that can refer to each application will be used below.

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represented by means of an oriented graph, like the one in the figure below.

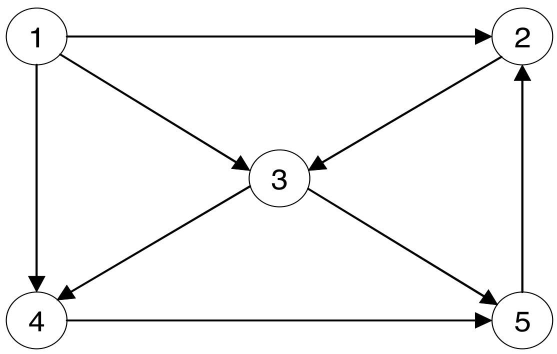

Figure 5.

In this simple explanation of the problem the 5 nodes of the oriented graph can be associated with the localities (intermodal centers, stations, depots, industrial plants, urban areas, etc.), and the 8 arcs with the communication routes which are supposed, for example, to one way (gas pipeline, road, railway branches, etc.) between the considered locations. In the real case of two-way streets, it will then be necessary, for each of them, to consider a pair of arcs directed in opposite directions between the two localities of reference. The network topology constitutes only a part of the problem data, to which must be added the demand for the goods, the availability of these in other locations, the transport costs from one location to another and the maximum capacity associated with each arc of the network (see example of integrated logistics network below [18]).

Figure 6.

To formulate this problem with graph theory (m variables and n constraints) we consider an oriented network $G=(N,L)$, associating to it for each arc $(i,j) \in L$ a cost C_{ij} (negative, positive or zero)①, a given capacity $C_{ij} \geq 0$ and a lower l_{ij} , with $0 \leq l_{ij} \leq C_{ij}$.

Suppose that each node i of N has an integer number $b(i)$, assuming that $b(i) > 0$ indicates the presence of an offer (i = origin, or "source") and that $b(i) < 0$ characterizes the presence of demand (i = destination, or "well"), while $b(i)$ =0 indicates the absence of both supply and demand (i= transfer node).

In the following we will assume the hypothesis (of admissibility) for which the total offer equals the total demand, given by the equation:

$$\sum_{i \in N} b(i) = 0.$$

Then, the cost of a flow can be defined as the sum:

$$\sum_{(i \to j) \in L} c_{ij} \cdot x_{ij} \cdot$$

It should be noted that the classification of the nodes in the three mentioned types (origin, destination, transfer or transit) is completely independent of the structure of the network, but it is defined only by the numerical data on the availability of the asset and demand. So, for example, in the previous scheme, if there is a demand for 6 units at node 4, 8 at node 5, and an availability of 10 units at node 1, and 4 at node 2, then 4 and 5 they are well nodes, while 1 and 2 are source nodes, while the 3 is transit.

Assuming now that the availability of the asset corresponds to a negative demand, it is possible to associate each node $i=1,..,n$ a variable equal to the demand b_i of the node (positive for the wells, negative for the sources and zero for the transit nodes). According to this approach, the application vector b will have n components, in which the i-th component will be b_i, definable by means of the expression:

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = \begin{pmatrix} -10 \\ -4 \\ 0 \\ 6 \\ 8 \end{pmatrix}.$$

Furthermore, for the calculation, we consider the hypothesis that the total availability of all the source nodes is equal to the total demand of all the well nodes which does not limit the applicability of the model to the practical problems concerning the transport of goods, because technically they can always be traced back to a form that satisfies this assumption. This, in fact, will be achieved by adding appropriate fictitious nodes and arcs in the network under study, with functions completely analogous to the waste variables used to bring the problems examined back to standard form.

① A unit of that property under study that is shipped from an "i" origin to a "j" destination must bear a cost equal to the sum of the unit transport costs of all the arcs of the network that must be crossed.

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In general, the problem of flow at minimum cost consists in to find an admissible flow distribution on the network $G=(N,L)$ such that the total cost is minimal. If we indicate the incidence matrix of the network with A_G with $b=[b(1),b(2),...,b(n)]^T$ the supply/demand vector, with R^m the cost vector and with $l \in (R^m)_+, u \in (R^m)_+$ the vectors whose components represent the lower limit and the capacity of the corresponding arc, the minimum cost flow problem can be analytically defined through the following system:

$$\begin{cases} \min c^T x \\ A_G x = b \\ l \le x \le u \end{cases} .$$

In the particular case of a non-capacitive network, that is when $l_{ij}=0$, $u_{ij}=\infty$, then the system assumes the following configuration:

$$\begin{cases} \min c^T x \\ A_G x = b \\ x \ge 0 \end{cases}$$

Finally, as regards, in general, for the resolution of any problem of optimization of the efficiency, quality, safety and reliability of a complex network[10, 11], the definition of the "objective function" (for the measurement of the value attributable to the set objective), this will depend on the number "n" of the choice variables $x_1,x_2,\ldots,x_n$, characterizing the case study (whose values must be established by the project manager, or by the multidisciplinary study team) and by a set of constraints that must be satisfied, expressed by equations or inequalities. These last can be "direct constraints", when they operate on the value of the variables of choice (for example, $1\le x_1\le 6$, "functional constraints", when they operate indirectly on them (for example $1\le x_1\le 6$ "functional constraints", when they operate indirectly on them (for example, $x_1+x_2\le 7$), or "non-negative constraints", when they impose on these variables to assume only non-negative values (for example, $x_3\ge 0$). If, we indicate the objective function by F, the maximum problem to be solved will be represented as follows:

$$\begin{cases} Max\{F(x_1,x_2,...x_n)\} \\ x_1,x_2,...x_n \\ g_1(x_1,x_2,...x_n) \le b_1 \\ g_2(x_1,x_2,...x_n) \le b_2 \\ \cdots\cdots\cdots\cdots \\ g_m(x_1,x_2,...x_n) \le b_m \end{cases}$$

where $F(x_1, x_2, ..., x_n)$ it is the function to be minimized (for the search for the lowest cost route for an O-D relationship on a mobility network) or to be maximized (if you want, for example, to examine the revenue from the toll, etc.), the variables $x_1, x_2, ... , x_n$ are those on which the operator can intervene and the expression $g_j(x_1, x_2, ..., x_n) \le b_j$ represent the m constraints (con j=1, 2, ... , m), of the problem to be satisfied, which here, by way of example, have all been expressed in the form of inequalities of the type [≤], that is, with a weak inequalities.

However, the objective function F varies according to the cases and could be of the first degree (linear), or of the higher degree (nonlinear), of the deterministic type, or random (with the presence of random variables); moreover, the problem could refer to a single temporal phase ("static" case), or more successive and related phases ("sequential" or "dynamic" problems), etc., and therefore in practical applications it will be necessary to use the procedure of more appropriate resolution and referring to appropriate computer programs.

Finally, assigning the "optimal" values thus identified to the choice variables, we will obtain the best possible result among all the admissible ones, interpreting the reality in the studio as closely as possible, in order to be able to provide the decision maker with useful tools to operate successful choices.

2. The Contribution of Stochastic Geometry to the Analysis of Safety, Reliability and Quality in Operation of a Complex Network

Associating a representative set of a network with a suitable set of variables, it is possible to carry out the functional analysis of the "network-safety-reliability-quality" system, through particular stochastic $\Re^n$ models that allow to calculate in the geometric space the interferences between the flow that crosses the branches of the network (vehicular flow, of fluids or gases, flow of information, relationships, etc.) and accidental obstacles placed along the path that can affect its normal operation, thus ensuring the best overall performance and system security.

The problem can therefore be addressed using a set of safety, reliability and quality indicators, $P_x^S(x), P_x^A(x), P_x^Q(x)$ which give rise to an indicator $P_x^G(x)$ of "global quality in operation" of the network, given by the expression:

$$P_x^G(x) = \sum\left[P_x^S(x)\right]^{-w_S} + \left[P_x^A(x)\right]^{-w_A} + \left[P_x^Q(x)\right]^{-w_Q},$$

where, case by case, the relative set of variables must be made explicit, including:

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(1) the overall costs[14, 15] of ris

$$C_r^R = \sum \left(C_{i \to j} \right)^{-w_{i \to j}}$$

for each arc $i \to j$ of the network (depending on the topology, peculiarities of the nodes and arcs, operating characteristics, level of tolerability of the risk for the user, emergency management, etc.);

(2) quality of the service, in terms of the generalized cost of transport, as the ratio between the minimum cost and the cost travel between the branches $i \to j$ of the network, which can be expressed with the ratio

$$Q = \sum \left(\frac{C_{min}^{i \to j}}{C_{eff}^{i \to j}} \right);$$

(3) the degree of interconnection of the n nodes

$$\beta = \sum \left(\beta_{i \to j}^{i=1,\dots,n} \right)^{-w_{\beta_i}};$$

(4) the level of intrinsic security of the infrastructure

$$L_S = \sum \left(L_{i \to j}^{i=1,\dots,n} \right)^{-w_{L_i}};$$

(5) reliability characteristics[10] in the useful life cycle of the technological devices in the network itineraries

$$T_{aff} = \sum \left(T_{i \to j}^{i=1,\dots,n} \right)^{-w_{T_i}};$$

(6) associating the concept of vulnerability[9] with that of functionality and the loss of functionality of certain arcs, a characteristic indicator of the network

$$\eta = \frac{df}{dv},$$

where f is the functionality expressed in terms of efficiency and quality of the offer and v represents the extension of the malfunction (also as calculated risk) for the entire network;

(7) topological efficiency of the network, as an estimate of the probability that all nodes are reachable, which if d_i indicates the shortest path between i and j can be represented by the expression

$$E_{Top}(G) = \frac{1}{N-1} \sum_{i,j \in G} \frac{1}{d_{i \to j}};$$

(8) *index of resilience*① [17], understood as "service recovery capacity" also following a disaster recovery *disaster ricovery*②, as an indicator of the system's ability to adapt to conditions of use and to resist wear and tear so as to continue to guarantee the availability of services provided (network fragility index), definable with the expression

$$(I_R)^{G=(N,\,L)} = \frac{\int_{t_1}^{t_2} \phi(t)\,dt}{(t_2 - t_1)} \cdot \theta,$$

where the function

$$\theta = f\left(k_i^{i=1,\dots,n}, E_{Top}, \beta \right),$$

depends on the variable E_{Top} and β defined above and on the total degree of each of the n nodes k_i of the network oriented graph, with

$$k_i = k_i^{out} + k_i^{in},$$

sum of the incoming and outgoing arches, being

$$k_i^{out} = \sum_{j \in N} a_{ij};\; k_i^{in} = \sum_{j \in N} a_{ji},$$

with a_{ji} the elements of the adjacency matrix. If the network is represented by a weighted graph, the term $k_i^{i=1,\dots,n}$ will represent the "node strength", defined as the sum of all the weights of the arches connected to node i, extending the mathematical relationship is to the incident arches, both to their weight.

Furthermore, $\phi(t)$ is a characteristic function of the system service built through the service level in an undisturbed situation ϕ_{ind}, the one following the event examined ϕ_a, the variable which measures the speed of recovery of the system and Δt=(t_2-t_1) the time interval considered. This function is given by the equation

$$\phi(t) = \phi_{ind} - \left(\phi_{ind} - \phi_o \right) \cdot e^{\varepsilon_r},$$

① The word resilience derives from the Latin verb "resilio", which means "to jump back, to return to a previous state". For a complex system we can distinguish the supply side resilience, or the functional resilience of the infrastructure, and the custumer side resilience, typical of the use of the network by its users. The issue of resilience takes on particular relevance, as it is mainly related to the "response" to unexpected events and sudden changes, such as in the case of accidental obstacles in the network.

② For computer networks, the term disaster recovery means the set of technological, logistical and organizational measures aimed at restoring data, systems and infrastructures necessary for the provision of business services to companies, associations or entities, in the face of serious emergencies that affect its regular activity.

and $\phi(t)$ can take values between 0 and 1, with $\phi(t)=1$ corresponding to the full operation of the system and $\phi(t)=0$ in the case of total inoperability. In the case of a network of n interdependent infrastructures and assuming that R_i is the resilience of the generic infrastructure *i-esima*, (i=1,...,n) it is then possible to identify a function $R_{tete}=\omega(R_1,...,R_i,...,R_n)$ which describes the resilience of the entire network.

(9) impact indicator on the functioning of the system for the measurement of the probability of interference of the flows with any obstacles along the route, depending of the network topology P_{int}. The characteristics of the "traffic" l_i^t, and those of the obstacles (assumed overall in number of "k") defined by the size d_k of each,

$$P_{int.} = f\ (l_i, d_k)\ .$$

This last aspect, since it assume the particular importance for the systematic analysis of a network system, will be further developed through a special stochastic model based on a lattice obtained starting from a geometric figure called the "fundamental cell", at the top of which are represented some regular polygons equal to each other, called "obstacles", obtaining a "regular lattice with obstacles".

Considering that in a complex network it is possible the simultaneous presence of multiple obstacles along the constituent arches, due to events that affect quality, safety and functionality, the case of the presence of a generic number equal to $2(n+2)$ of obstacles.

The proposed model is, among other things, functional for the management of seismic and atmospheric emergencies for all physical transport networks and is also applicable to intangible ones.

In fact, in the studies of the infrastructures represented by graphs in $\Re^n$, the tool of geometric probabilities lends itself well to the analysis of interferences in operation of the relationship "network-obstacles-safety-system operation", assuming that the entire network is formed by a union union R of elementary geometric figures, constituting special lattices R_i, whereby $R=R_1,R_2,...,R_n$.

3. A Mathematical Model for Assessing the Probability of Interference of the Flow with Multiple Accidental Obstacles Along a Network System

Considering the graph $G=(N,L)$ representative of the considered network and constructed the sub-latticeolo R_i of the union set formed by it, by means of particular test bodies (mathematical models) representative of the means constituting the flow that passes through it, it is possible to study the relative motion in R_i(i=1,...,n) and any interferences generated by obstacles (of fixed shape and size) along each arc $i\rightarrow j$ of the network[12].

These test bodies can be taken as segments of suitable length l (to schematize, for example, in a road network a freight train with a large number of carriages, or an articulated lorry in the case of road transport), or consisting of rectangles of sides l_1 and l_2 (as in the case of a high-speed train, or a car along a road).

Furthermore, in order to research the mathematical solution of the problem, we will hypothesize that each side of the considered reticle offers the same resistance to the forward movement of the test body and that there are conditions of uniform motion.

We consider a lattice with the fundamental cell a rectangle (but it could be any other flat figure, even irregular, constituting the lattice) in which three different types of obstacles are inserted, in an equal number up to $2(n+2)$, formed by the following geometric figures: rhombuses, circular sectors and square.The degree of incidence of the disturbance produced by them on the regularity of the outflow in the network concerned varies from the maximum impact of the rhombuses to the minimum impact of the circular sectors, which have the least probability of interference between the test body (medium) and long obstacle path.

A reference scheme for the practical applications of the method is shown in the following table and refers both to the study of tangible and intangible networks, such as web networks, where for anomalies from accidental events, such as the occurrence of a "Bottleneck", one must evaluate, in analogy to the electrical circuits, the impedance①, that is the resistance opposed by the obstacle to the flow of information. The same approach can be considered in the case of a mobility or logistics system for the analysis of traffic safety and fluidity, or in emergency landslides for adverse weather situations, monitoring the problem of the probability of obstacle-vehicle interference (fall boulders, trees, etc. along the infrastructure network or its nodes).

① Impedance, in electrical engineering, is a physical quantity that represents the opposition force of a circuit to the passage of an alternating electric current, or, more generally, of a variable current.

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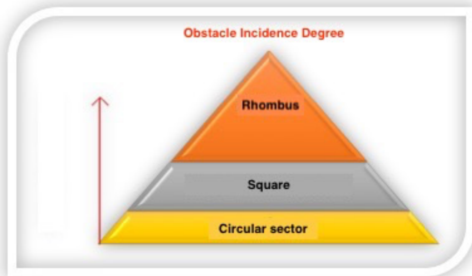


Figure 7.

Let $\Re(a,b,c)$ be the regular lattice with fundamental cell is as in Figure 8.

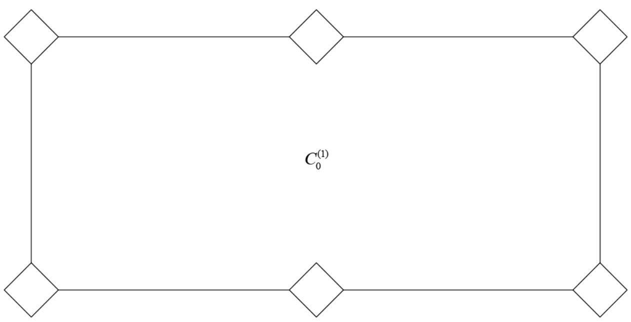


Figure 8.

Denoting with $C_0^{(1)}$ the fundamental cell, we have:

$$(1)\ area\, C_0^{(1)} = 2ab - c^2.$$

The cell $C_0^{(1)}$ have six obstacles that are squares with diagonal of length c with $c<min(a,b)$. Considering a segment s of random position and of length l with $c<l<min(a,b)$, we want compute the probability that this segment intersects a side of lattice. This probability is equal to probability $P_{\text{int}}^{(1)}$ that the segment s intersects the boundary of the fundamental cell. The position of the segment s is determined by the middle point p and by the angle φ that the segment form with the axis x. We consider the limit positions of the segment s that corresponds at angle φ and let $\hat{C}_0^{(1)}(\varphi)$ the determined figure from this position (see Figure 9):

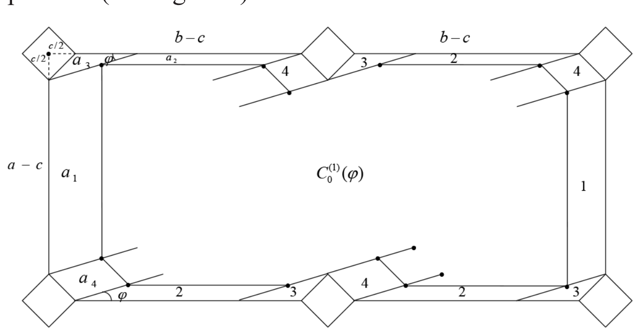


Figure 9.

Considering some results that we have obtained in a previous paper[1], and from Figure 9 follow that:

$$area\, a_1(\phi) = \frac{(a-c)}{2} l \cos\phi,$$

$$area\left[a_2(\phi) + a_3(\phi)\right] = \frac{(b-c)l^2}{2} \sin\phi,$$

$$area\, a_4(\phi) = \frac{cl}{4}(\sin\phi + 2\cos\phi) - \frac{l^2}{4}\sin 2\phi,$$

we have that:

$$area\, a_4(\phi) = \frac{cl}{4}\left(\sin\phi + \frac{3}{2}\cos\phi\right) - \frac{l^2}{8}\sin 2\phi.$$

With these results the relation (1) give us:

(2)

$$area\, \hat{C}_0^{(1)}(\phi) = 2ab - c^2$$
$$-\left[(a-c)l\cos\phi + 2(b-c)l\sin\phi + cl\left(\sin\phi + \frac{3}{2}\cos\phi\right) - \frac{l^2}{2}\sin 2\phi\right]$$
$$= 2ab - c^2 - \left[\left(a - \frac{c}{2}\right)l\cos\phi + (2b-c)l\sin\phi - \frac{l^2}{2}\sin 2\phi\right].$$

Denoting with M, the set of segments s whose the middle point are in $C_0^{(1)}$ and N, the set of segments s whose the middle point are in $\hat{C}_0^{(1)}$, we have that:

$$(3)\ P_{\text{int}}^{(1)} = 1 - \frac{\mu(N_1)}{\mu(M_1)}$$

where μ is the Lebesgue measure in Euclidean plane[4]. In order to compute the measures $\mu(M,)$ and $\mu(N,)$ we use the Poincaré kinematic measure[3] $dK = dx \wedge dy \wedge d\phi$,

where x,y are the coordinates of p and φ the defined angle. Since $\varphi \in [0, (\pi/2)]$, we obtain that:

(4)

$$\mu(M_1) = \int_{\alpha}^{\frac{\pi}{2}} d\phi = \iint_{\left\{(x,y)\in C_0^{(1)}\right\}} dxdy = \frac{\pi}{2} area C_0^{(1)} = \frac{\pi}{2}\left(2ab - c^2\right),$$

and considering the (2)

(5)

$$\mu(N_1) = \int_{\alpha}^{\frac{\pi}{2}} d\phi = \iint_{\left\{(x,y)\in \hat{C}_0^{(1)}\right\}} dxdy = \int_{\alpha}^{\frac{\pi}{2}} \left[area\, C_0^{(1)}(\phi)\right] d\phi =$$

$$\frac{\pi}{2}\left(2ab-c^2\right)-\int_{\alpha}^{\frac{\pi}{2}}\left[\left(a+\frac{c}{2}\right)l\cos\phi+\left(2b-c\right)l\sin\phi-\frac{l^2}{2}\sin 2\phi\right]d\phi=$$

$$\frac{\pi}{2}\left(2ab-c^2\right)-\left[\left(a+2b-\frac{c}{2}\right)l-\frac{l^2}{2}\right].$$

Considering the (3), (4) and (5) we obtain the probability:

$$(6)\quad P_{\text{int}}^{(1)}=\frac{2\left(a+2b-\frac{c}{2}\right)l-l^2}{\pi\left(2ab-c^2\right)}.$$

When $c\to 0$, the obstacles becomes points and the fundamental cell becomes a rectangle with side a and $2b$. In this case the probability (6) becomes the Laplace probability:

$$P=\frac{2\left(a+2b\right)l-l^2}{2\pi ab}.$$

In the same way we can consider other different lattice configurations.

Example 1. Let $\Re_2$(a,b,c) be the regular lattice with fundamental cell is as in Figure 10:

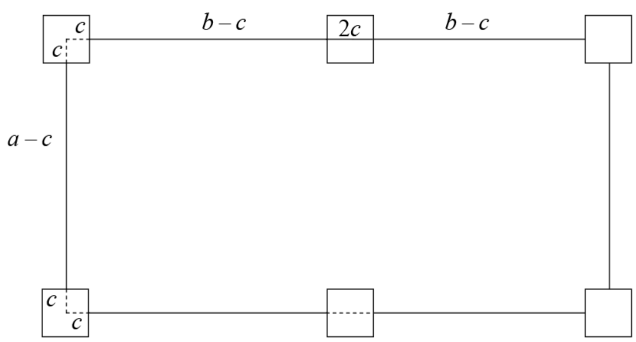


Figure 10.

The obstacles are squares with the side of length c. Considering a random segment whose length $l<min(a,b)$ we to want compute the probability that a segment intersects a side of the lattice. This probability is equal to the probability $P_{\text{int}}^{(2)}$ that a segment s intersect a side of the fundamental cell $C_0^{(2)}$ we have

$$area\, C_0^{(2)}=2ab-8\cdot\frac{c^2}{8}=2ab-c^2$$

Considering the position s of the segment with barycenter p and which forms an angle φ with axis x. We consider the limit positions of the segment s for a determined angle φ, and let $\widehat{C}_0^{(2)}(\varphi)$ the figure determined by these positions (see Figure 11):

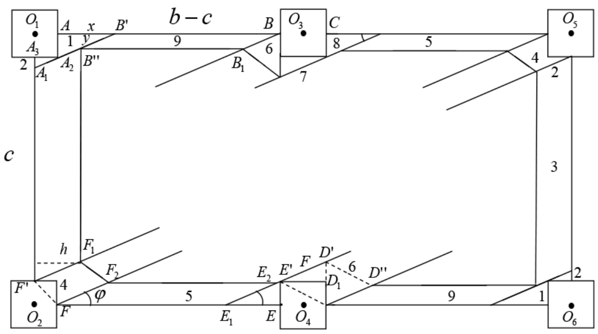


Figure 11.

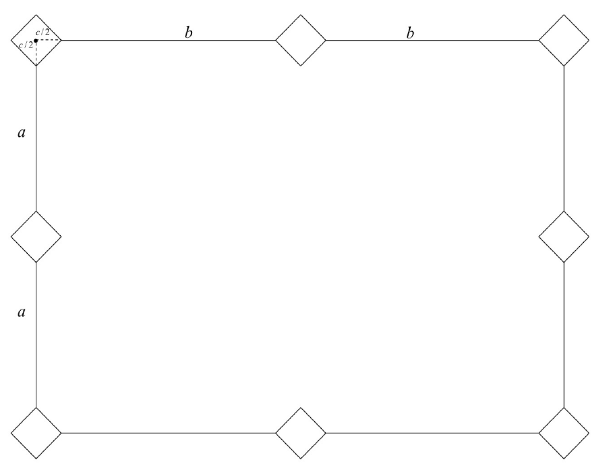
and we have that

$$P_{\text{int}}^{(2)}=\frac{l\left(a+2b-\frac{c}{4}\right)-\frac{l^2}{2}-\pi c^2}{\frac{\pi}{2}\left(2ab-c^2\right)}=\frac{2l\left(a+2b-\frac{c}{4}\right)-l^2-2\pi c^2}{\pi\left(2ab-c^2\right)}$$

When $c\to 0$ the fundamental cell $C_0^{(2)}$ becomes a rectangle with sides a and $2b$ and the obstacles become points and the probability $P_{\text{int}}^{(2)}$ becomes the Laplace probability:

$$P=\frac{2\left(a+2b\right)-l^2}{2\pi ab}$$

Example 2. Let $\Re_2$(a,b,c) be the regular lattice with fundamental cell is as in Figure 12.



f.

Figure 12.

Denoting with $C_0^{(3)}$ the fundamental cell, we have:

$$area\, C_0^{(3)}=4ab-\frac{3c^2}{2}$$

The cell $C_0^{(3)}$ have eight obstacles that are squares with

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diagonal of length $2c$ with $c<min(a,b)$. We have

$$(7)\ P_{\text{int}}^{(3)} = \frac{2\left(2a+2b-\frac{c}{2}\right)l-l^2}{\pi\left(4ab-\frac{3c^2}{2}\right)}$$

When $c\to 0$, the obstacles becomes points and the fundamental cell becomes a rectangle with side $2a$ and $2b$. In this case the probability (7) becomes the Laplace probability:

$$P_{\text{int}}^{(3)} = \frac{2(2a+2b)l-l^2}{4\pi ab}$$

Example 3. Let $\Re_4(a,b,c;\alpha)$ be the regular lattice with fundamental cell $C_0^{(4)}$ is as in Figure 13.

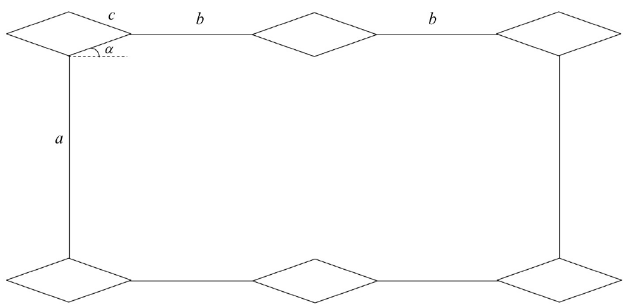

Figure 13.

where $\alpha \in [0, (\pi/2)]$ is an angle and $c<min(a,b)$. The $C_0^{(4)}$ have six obstacles that are rhombs with side c and with the diagonals $d_1=2c\sin\alpha$, $d_2=2c\cos\alpha$. We have that:

$$area\, C_0^{(4)} = 2ab + 4c(\alpha\cos\alpha + b\sin\alpha) + 4c^2\sin\alpha\cos\alpha$$

We have

$$(8)\ P_{\text{int}}^{(4)} = \frac{2(a+2b+4c\cos\alpha)l-l^2}{\pi\left[2ab+4c(a\cos\alpha+b\sin\alpha)+4c^2\sin\alpha\cos\alpha\right]}.$$

When $\alpha\to 0$, the obstacles becomes segments of length $2c$ that go in the boundary of the lattice, the fundamental cell $C_0^{(4)}$ becomes a rectangle of side a and $2b+4c$ and the (8) becomes the Laplace probability:

$$P = \frac{2(a+2b+4c)l-l^2}{\pi a(2b+4c)}.$$

Example 4. Let $\Re_5(a,b,c;n)$ be the regular lattice with fundamental cell $C_0^{(n)}$ a rectangle with sides $(n+1)\, a$ and b with $2(n+2)$ obstacles that are four quarter of circle of radius $(c/2)$ and 2n semicircle with same radius $(c/2)$ (Figure 14):

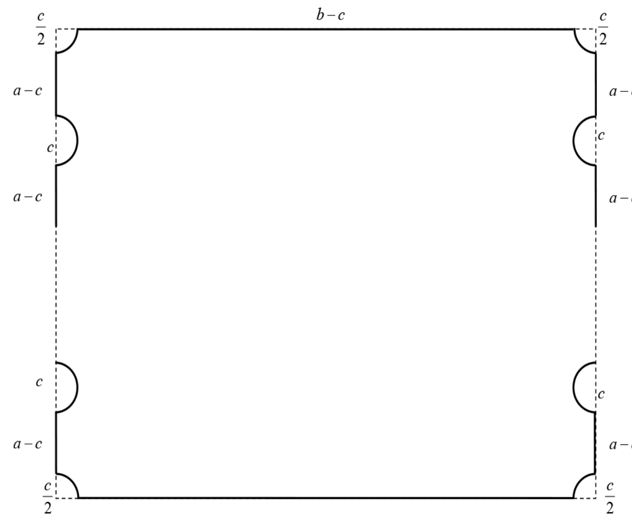

Figure 14.

We have that:

(9)

$$P_{\text{int}}^{(n)} = \frac{2\left[(n+1)a+b-c\right]l-l^2-\frac{\pi\left[(n+1)\pi-1\right]c^2}{4}}{\pi\left[(n+1)ab-\frac{(n+1)\pi c^2}{4}\right]} s.$$

When $c\to 0$, the fundamental cell $C_0^{(n)}$ becomes a rectangle of sides $(n+1)$a and b and the probability (9) becomes the Laplace probability:

$$P_{\text{int}}^{(n)} = \frac{2\left[(n+1)a+b\right]l-l^2}{\pi(n+1)ab}$$

4. Conclusion

In this work, a particular mathematical criterion has been proposed for the analysis of the performance characteristics and reliability in operation of the material and immaterial[20] networks in the presence of accidental obstacles, through the study of geometric probabilities and integral geometry. It covers a wide range of engineering, socio-economic and applied sciences applications, being able to deal with problems of Social Network Analysis, mobility and logistic systems(5), business, etc. In fact, the algorithm allows to evaluate, with an adequate degree of accuracy, the interferences of the "network - unexpected obstacles - interferences with the flow of traffic" system, representing the network with an appropriate regular grid formed by the union of equal regular polygons,

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generated by a "fundamental cell" and which have only segments of the relative "borders" in common.

For the development of the model, other equal regular polygons, called "obstacles", were positioned in the vertices of the fundamental cell, thus geometrically constructing a particular "regular grid with obstacles", capable of defining the ordinary operating conditions of the network and to calculate the probability of "flow-obstacles" interference for each O-D path (origin-destination).

In this way, it is possible to study the effects of an accidental "disturbing cause" on the circulation of the "vehicles" that make up the current of traffic and also to intervene to restore the functionality of the system. To fully define the problem, the "impacts" on the lattice of three different types of obstacles (square, circular sector and rhombus) were examined, considering a multiple number, up to 2(n+2). This, in order to appropriately represent the different degree of incidence of the disturbance on the network (min circular sector; max rhombus) according to the different unforeseen causes that can temporarily alter its regular performance characteristics and those of safety and reliability. For display simplicity, a rectangle has been assumed here as a "cell", obviously being able to extend the calculation (also with the aid of computerized processing) to any other flat, regular and / or irregular figure, constructing the representative grid of any case concrete through the effective articulation of arcs and nodes in the development of the network.

In this sense, by way of example, the configurations of both the internet network and the national road network ANAS complete with its TEN (Trans-European Networks) extensions are shown below.

Figure 15.

Figure 16.

In the case of complex networks such as those shown above, it is necessary to extend the results reported in the present work to the actual configurations in R^n, also considering the rotation of the test body formed by the segment s representative of the medium belonging to the traffic current flowing into the network (material or immaterial).

That is, it is necessary to take into account the particular geometric configuration of the lattice to be examined, considering in the calculation of the geometric probabilities developed above also the variable φ_0 given by the value assumed by the angle between the $0x$ axis and the straight support d of the aforementioned segment s .

If $P_m=(x,y)$ is the midpoint of s (of length l) and x and y represent its Cartesian coordinates, the problem can therefore be solved, in the general case, considering the elementary kinematic measure of Poincaré in the Euclidean plane [17]:

$$dk = dx \wedge dy \wedge d\varphi_0 .$$

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