Journal of Computer Science Research
https://ojs.bilpublishing.com/index.php/jcsr

ARTICLE

Periodic Solution for a Complex-Valued Network Model with Discrete Delay

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ARTICLE INFO

Article history
Received: 21 January 2022
Accepted: 16 February 2022
Published Online: 28 February 2022

Keywords:
Complex-valued neural network model
Delay
Periodic solution

ABSTRACT

For a tridiagonal two-layer real six-neuron model, the Hopf bifurcation was investigated by studying the eigenvalue equations of the related linear system in the literature. In the present paper, we extend this two-layer real six-neuron network model into a complex-valued delayed network model. Based on the mathematical analysis method, some sufficient conditions to guarantee the existence of periodic oscillatory solutions are established under the assumption that the activation function can be separated into its real and imaginary parts. Our sufficient conditions obtained by the mathematical analysis method in this paper are simpler than those obtained by the Hopf bifurcation method. Computer simulation is provided to illustrate the correctness of the theoretical results.

1. Introduction

Recently, various complex-valued network models with or without time delays have been studied [1-4,6-20]. For example, Ji et al. have investigated the following complex-valued Wilson-Cowan neural network model:

$$\begin{aligned} w_1'(t) &= -w_1(t) + a_1 g\big(w_1(t)\big) + a_2 g\big(w_2(t-\tau)\big) + P \\ w_2'(t) &= -w_2(t) + a_3 g\big(w_1(t-\tau)\big) + a_4 g\big(w_2(t)\big) + Q \end{aligned} \tag{1}$$

By using proper translations and coordinate transformations, system (1) has been decomposed the functions $g(w_1)$, $g(w_2)$ and a_1, a_2, a_3, a_4 into their real and imaginary parts, thus an equivalent real-valued system has been constructed. Then, the sufficient conditions for the Hopf bifurcation and its directions were provided [1]. Hang et al. have investigated a two-node network system as follows [2]:

$$\begin{aligned} D^q z_1(t) &= -\mu_1 z_1(t) + af\big(z_1(t-\tau)\big) + bf\big(z_2(t-\tau)\big) \\ D^q z_2(t) &= -\mu_2 z_2(t) + cf\big(z_1(t-\tau)\big) + df\big(z_2(t-\tau)\big) \end{aligned} \tag{2}$$

About the dynamical behaviors, local asymptotical stability and the Hopf bifurcation were studied, the important conditions of emergence of bifurcation were also given. Li et al. [3]extended a real-valued network model into a complex-valued model as the following:

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DOI: https://doi.org/10.30564/jcsr.v4i1.4374

$$\begin{aligned} z_1'(t) &= -z_1(t) + b_{11}f_{11}\left(z_1(t-\tau)\right) + b_{12}f_{12}\left(\int_{-\infty}^{t} F(t-s)\,z_2(s)\,ds\right) \\ z_2'(t) &= -z_2(t) + b_{21}f_{21}\left(z_1(t-\tau)\right) + b_{22}f_{22}\left(\int_{-\infty}^{t} F(t-s)\,z_2(s)\,ds\right) \end{aligned} \tag{3}$$

Regarding the discrete time delay as the bifurcating parameter, the problem of the Hopf bifurcation in the newly-proposed complex-valued neural network model was investigated under the assumption that the activation function can be separated into its real and imaginary parts. Based on the normal form theory and center manifold theorem, some sufficient conditions which determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions were established. Zhang et al. have considered a complex value delayed Hopfield neural networks model [4]:

$$\begin{aligned} z_1'(t) &= (a+ib)z_1(t) + (c+id)f\left(z_2(t)\right) + (m+in)f\left(w_1(t-\tau)\right) \\ z_2'(t) &= (a+ib)z_2(t) + (c+id)f\left(z_3(t)\right) + (m+in)f\left(w_2(t-\tau)\right) \\ z_3'(t) &= (a+ib)z_3(t) + (c+id)f\left(z_4(t)\right) + (m+in)f\left(w_3(t-\tau)\right) \\ z_4'(t) &= (a+ib)z_4(t) + (c+id)f\left(z_1(t)\right) + (m+in)f\left(w_4(t-\tau)\right) \\ w_1'(t) &= (a+ib)w_1(t) + (c+id)f\left(w_2(t)\right) + (m+in)f\left(z_1(t-\tau)\right) \\ w_2'(t) &= (a+ib)w_2(t) + (c+id)f\left(w_3(t)\right) + (m+in)f\left(z_2(t-\tau)\right) \\ w_3'(t) &= (a+ib)w_3(t) + (c+id)f\left(w_4(t)\right) + (m+in)f\left(z_3(t-\tau)\right) \\ w_4'(t) &= (a+ib)w_4(t) + (c+id)f\left(w_1(t)\right) + (m+in)f\left(z_4(t-\tau)\right) \end{aligned} \tag{4}$$

By using the basic bifurcation theory of delay differential equations, and the theory of Lie groups, the authors have discussed the bifurcating periodic solutions. The existence of multiple branches of the bifurcating periodic solution was also provided.

In this paper, we extend a real six-neuron network [5] to the following complex-valued model:

$$\begin{aligned} z_1'(t) &= -(a_1+ib_1)z_1(t) + (m_{14}+in_{14})f_{14}\left(z_4(t-\tau)\right) + (m_{15}+in_{15})f_{15}\left(z_5(t-\tau)\right) \\ z_2'(t) &= -(a_2+ib_2)z_2(t) + (m_{24}+in_{24})f_{24}\left(z_4(t-\tau)\right) + (m_{25}+in_{25})f_{25}\left(z_5(t-\tau)\right) \\ &\quad + (m_{26}+in_{26})f_{26}\left(z_6(t-\tau)\right) \\ z_3'(t) &= -(a_3+ib_3)z_3(t) + (m_{35}+in_{35})f_{35}\left(z_5(t-\tau)\right) + (m_{36}+in_{36})f_{36}\left(z_6(t-\tau)\right) \\ z_4'(t) &= -(a_4+ib_4)z_4(t) + (m_{41}+in_{41})f_{41}\left(z_1(t-\tau)\right) + (m_{42}+in_{42})f_{42}\left(z_2(t-\tau)\right) \\ z_5'(t) &= -(a_5+ib_5)z_5(t) + (m_{51}+in_{51})f_{51}\left(z_1(t-\tau)\right) + (m_{52}+in_{52})f_{52}\left(z_2(t-\tau)\right) \\ &\quad + (m_{53}+in_{53})f_{53}\left(z_3(t-\tau)\right) \\ z_6'(t) &= -(a_6+ib_6)z_6(t) + (m_{62}+in_{62})f_{62}\left(z_2(t-\tau)\right) + (m_{63}+in_{63})f_{63}\left(z_3(t-\tau)\right) \end{aligned} \tag{5}$$

where $z_j = x_j + iy_j, a_j, b_j, m_{kj}, n_{kj}$ are real numbers, f_{kj} are activation functions, $k, j=1, 2, \ldots, 6$.We will discuss the dynamic behavior of the solutions of system (5).

We point out that the bifurcating method is not easy to deal with system (5) if all a_j, b_j, m_{kj}, n_{kj} are different real numbers. In this paper, by means of the mathematical analysis method, we discuss the periodic oscillation for system (5). For convenience, let $f_{kj}\left(z_j(t-\tau)\right) = f_{kj}^{R}\left(x_j(t-\tau), y_j(t-\tau)\right) + if_{kj}^{I}\left(x_j(t-\tau), y_j(t-\tau)\right) = f_{kj}^{R} + f_{kj}^{I}$ ($z_j = x_j + iy_j$, $k, j=1, 2, \ldots, 6$). Then the complex-valued system (5) can be expressed by separating it into real and imaginary parts as the following:

$$\begin{aligned} x_1'(t) &= -a_1x_1(t) + b_1y_1(t) + m_{14}f_{14}^{R} - n_{14}f_{14}^{I} + m_{15}f_{15}^{R} - n_{15}f_{15}^{I} \\ y_1'(t) &= -b_1x_1(t) - a_1y_1(t) + n_{14}f_{14}^{R} + m_{14}f_{14}^{I} + n_{15}f_{15}^{R} + m_{15}f_{15}^{I} \\ x_2'(t) &= -a_2x_2(t) + b_2y_2(t) + m_{24}f_{24}^{R} - n_{24}f_{24}^{I} + m_{25}f_{25}^{R} - n_{25}f_{25}^{I} + m_{26}f_{26}^{R} - n_{26}f_{26}^{I} \\ y_2'(t) &= -b_2x_2(t) - a_2y_2(t) + n_{24}f_{24}^{R} + m_{24}f_{24}^{I} + n_{25}f_{25}^{R} + m_{25}f_{25}^{I} + n_{26}f_{26}^{R} + m_{26}f_{26}^{I} \\ x_3'(t) &= -a_3x_3(t) + b_3y_3(t) + m_{35}f_{35}^{R} - n_{35}f_{35}^{I} + m_{36}f_{36}^{R} - n_{36}f_{36}^{I} \\ y_3'(t) &= -b_3x_3(t) - a_3y_3(t) + n_{35}f_{35}^{R} + m_{35}f_{35}^{I} + n_{36}f_{36}^{R} + m_{36}f_{36}^{I} \\ x_4'(t) &= -a_4x_4(t) + b_4y_4(t) + m_{41}f_{41}^{R} - n_{41}f_{41}^{I} + m_{42}f_{42}^{R} - n_{42}f_{42}^{I} \\ y_4'(t) &= -b_4x_4(t) - a_4y_4(t) + n_{41}f_{41}^{R} + m_{41}f_{35}^{I} + n_{42}f_{42}^{R} + m_{42}f_{42}^{I} \\ x_5'(t) &= -a_5x_5(t) + b_5y_5(t) + m_{51}f_{51}^{R} - n_{51}f_{51}^{I} + m_{52}f_{52}^{R} - n_{52}f_{52}^{I} + m_{53}f_{53}^{R} - n_{53}f_{53}^{I} \\ y_5'(t) &= -b_5x_5(t) - a_5y_5(t) + n_{51}f_{51}^{R} + m_{51}f_{51}^{I} + n_{52}f_{52}^{R} + m_{52}f_{25}^{I} + n_{53}f_{53}^{R} + m_{53}f_{53}^{I} \\ x_6'(t) &= -a_6x_6(t) + b_6y_6(t) + m_{62}f_{62}^{R} - n_{62}f_{62}^{I} + m_{63}f_{63}^{R} - n_{63}f_{63}^{I} \\ y_6'(t) &= -b_6x_6(t) - a_6y_6(t) + n_{62}f_{62}^{R} + m_{62}f_{62}^{I} + n_{63}f_{63}^{R} + m_{63}f_{63}^{I} \end{aligned} \tag{6}$$

Therefore, in order to discuss the periodic solution of model (5), we only consider the periodic solution of system (6). Suppose that the derivative of $f_{kj}^{R}\left(x_j, y_j\right)$ $f_{kj}^{R}\left(x_j, y_j\right)$ with respect to x_j and y_j exist, continuous, and $f_{kj}^{R}(0,0) = 0$, $f_{kj}^{I}(0,0) = 0$. Then the linearized system of (6) is the following:

$$\begin{aligned} x_1'(t) &= -a_1x_1(t) + b_1y_1(t) + p_{14}x_4(t_\tau) + q_{14}y_4(t_\tau) + p_{15}x_5(t_\tau) + q_{15}y_5(t_\tau) \\ y_1'(t) &= -b_1x_1(t) - a_1y_1(t) + c_{14}x_4(t_\tau) + d_{14}y_4(t_\tau) + c_{15}x_5(t_\tau) + d_{15}y_5(t_\tau) \\ x_2'(t) &= -a_2x_2(t) + b_2y_2(t) + p_{24}x_4(t_\tau) + q_{24}y_4(t_\tau) + p_{25}x_5(t_\tau) + q_{25}y_5(t_\tau) \\ &\quad + p_{26}x_6(t_\tau) + q_{26}y_6(t_\tau) \\ y_2'(t) &= -b_2x_2(t) - a_2y_2(t) + c_{24}x_4(t_\tau) + d_{24}y_4(t_\tau) + c_{25}x_5(t_\tau) + d_{25}y_5(t_\tau) \\ &\quad + c_{26}x_6(t_\tau) + d_{26}y_6(t_\tau) \\ x_3'(t) &= -a_3x_3(t) + b_3y_3(t) + p_{35}x_5(t_\tau) + q_{35}y_5(t_\tau) + p_{36}x_6(t_\tau) + q_{36}y_6(t_\tau) \\ y_3'(t) &= -b_3x_3(t) - a_3y_3(t) + c_{35}x_5(t_\tau) + d_{35}y_5(t_\tau) + c_{36}x_6(t_\tau) + d_{36}y_6(t_\tau) \\ x_4'(t) &= -a_4x_4(t) + b_4y_4(t) + p_{41}x_1(t_\tau) + q_{41}y_1(t_\tau) + p_{42}x_2(t_\tau) + q_{42}y_2(t_\tau) \\ y_4'(t) &= -b_4x_4(t) - a_4y_4(t) + c_{41}x_1(t_\tau) + d_{41}y_1(t_\tau) + c_{42}x_2(t_\tau) + d_{42}y_2(t_\tau) \\ x_5'(t) &= -a_5x_5(t) + b_5y_5(t) + p_{51}x_1(t_\tau) + q_{51}y_1(t_\tau) + p_{52}x_2(t_\tau) + q_{52}y_2(t_\tau) \\ &\quad + p_{53}x_3(t_\tau) + q_{53}y_3(t_\tau) \\ y_5'(t) &= -b_5x_5(t) - a_5y_5(t) + c_{51}x_1(t_\tau) + d_{51}y_1(t_\tau) + c_{52}x_2(t_\tau) + d_{52}y_2(t_\tau) \\ &\quad + c_{53}x_3(t_\tau) + d_{53}y_3(t_\tau) \\ x_6'(t) &= -a_6x_6(t) + b_6y_6(t) + p_{62}x_2(t_\tau) + q_{62}y_2(t_\tau) + p_{63}x_3(t_\tau) + q_{63}y_3(t_\tau) \\ y_6'(t) &= -b_6x_6(t) - a_6y_6(t) + c_{62}x_2(t_\tau) + d_{62}y_2(t_\tau) + c_{63}x_3(t_\tau) + d_{63}y_3(t_\tau) \end{aligned} \tag{7}$$

where $x_j(t_\tau) = x_j(t-\tau)$, $y_j(t_\tau) = y_j(t-\tau)$, $p_{kj} = m_{kj}\dfrac{\partial f_{kj}^{R}(0,0)}{\partial x_j} - n_{kj}\dfrac{\partial f_{kj}^{I}(0,0)}{\partial x_j}$, $q_{kj} = m_{kj}\dfrac{\partial f_{kj}^{R}(0,0)}{\partial y_j} - n_{kj}\dfrac{\partial f_{kj}^{I}(0,0)}{\partial y_j}$, $c_{kj} = n_{kj}\dfrac{\partial f_{kj}^{I}(0,0)}{\partial x_j} + m_{kj}\dfrac{\partial f_{kj}^{R}(0,0)}{\partial x_j}$, $d_{kj} = n_{kj}\dfrac{\partial f_{kj}^{R}(0,0)}{\partial y_j} + m_{kj}\dfrac{\partial f_{kj}^{R}(0,0)}{\partial x_j}$.

The matrix form of system (7) is the following:

$$U'(t) = AU(t) + BU(t-\tau) \tag{8}$$

where $U(t) = [x_1(t), y_1(t), \ldots, x_6(t), y_6(t)]^T$, $U(t) = [x_1(t-\tau), y_1(t-\tau), \ldots, x_6(t-\tau), y_6(t-\tau)]^T$. Both A and B are 12 by 12 matrices as follows:

$$A=(a_{ij})_{12\times 12}=\begin{pmatrix} -a_1 & b_1 & & 0 & \cdots & 0 & 0 & 0 \\ -b_1 & -a_1 & & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -a_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -a_5 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -a_6 & b_6 \\ 0 & 0 & 0 & \cdots & 0 & -b_6 & -a_6 \end{pmatrix}$$

$$B=(b_{ij})_{12\times 12}=\begin{pmatrix} 0 & 0 & & 0 & \cdots & q_{15} & 0 & 0 \\ 0 & 0 & & 0 & \cdots & d_{15} & 0 & 0 \\ 0 & 0 & 0 & \cdots & q_{25} & p_{26} & q_{26} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{51} & d_{51} & c_{52} & \cdots & 0 & 0 & 0 \\ 0 & 0 & p_{62} & \cdots & 0 & 0 & 0 \\ 0 & 0 & c_{62} & \cdots & 0 & 0 & 0 \end{pmatrix}$$

2. Preliminaries

Lemma 1 Assume that $a_j>0,\ b_j>0, f^R_{kj}(0,0)=0, f^I_{kj}(0,0)=0, f^R_{kj}(x_j,y_j)>0,\ f^I_{kj}(x_j,y_j)>0$ when $x_j>0,\ y_j>0,$ while $f^R_{kj}(x_j,y_j)<0,\ f^I_{kj}(x_j,y_j)<0$ when $x_j<0,\ y_j<0\ (k,\ j=1,\ 2,\ \ldots,\ 6)$, $C=A+B$ is a nonsingular matrix, then system (6) has a unique equilibrium.

Proof An equilibrium $U^*=[x^*_1,\ y^*_1,\ \ldots,\ x^*_6,\ y^*_6]^T$ of system (8) is a constant solution of the following algebraic equation:

$$AU^*+BU^*=CU^*=0 \tag{9}$$

Since $C=A+B$ is a nonsingular matrix, then system (9) only has zero solution according to the linear algebra basic theorem. Noting that $f^R_{kj}(0,0)=0, f^I_{kj}(0,0)=0$ ($k, j=1,2,\ldots,6$). Therefore, zero is a solution of system (6). Obviously, zero is the unique equilibrium of system (6) since $f^R_{kj}(x_j,y_j)>0,\ f^I_{kj}(x_j,y_j)>0$ when $x_j>0,\ y_j>0,$ while $f^R_{kj}(x_j,y_j)<0,\ f^I_{kj}(x_j,y_j)<0$ when $x_j<0,\ y_j<0\ (k,\ j=1,\ 2,\ \ldots,\ 6)$.

Lemma 2 Assume that $f^R_{kj}(x_j,y_j)$, $f^I_{kj}(x_j,y_j)$ ($k, j=1,2,\ldots,6$) are continuous bounded functions, $a_j>0,\ b_j>0$ $(k,\ j=1,\ 2,\ \ldots,\ 6)$. Then all solutions of system (6) are uniformly bounded.

Proof Since $f^R_{kj}(x_j,y_j)$, $f^I_{kj}(x_j,y_j)$ ($k, j=1,2,\ldots,6$) are continuous bounded functions, then from system (6) we have

$$\begin{aligned} x'_j(t) &\le -a_jx_j(t)+b_jy_j(t)+N_{1j} \\ y'_j(t) &\le -b_jx_j(t)-a_jy_j(t)+N_{2j} \end{aligned} \tag{10}$$

where N_{1j}, N_{2j} are some positive constants. It is easily to see that all solutions of system (10) are uniformly bounded since $a_j>0,\ b_j>0\ (k,\ j=1,\ 2,\ \ldots,\ 6)$, implying that all solutions of system (6) are uniformly bounded.

3. Main Results

It is known that the instability of the trivial solution of system (7) guarantees the instability of the trivial solution of system (6). Thus, we have the following theorems.

Theorem 1 Assume that Lemma 1 and Lemma 2 hold for selecting parameter values of $a_j,\ b_j, m_{kj},\ n_{kj}(k,\ j=1,\ 2,\ \ldots,\ 6)$. Let the eigenvalues of matrices A, B be α_j and β_j $(j=1,\ 2,\ \ldots,\ 12)$ respectively. If there exists at least one eigenvalue β_k $(k\in\{1,\ 2,\ \ldots,\ 12\})$ such that $\beta_k>0$, or $Re(\beta_k)>\alpha_k$, where $\alpha_k=\min_{1\le j\le 6}\left|-a_j\right|$. Then system (6) generates a periodic oscillatory solution.

Proof Obviously, we only need to consider the instability of the trivial solution of system (7). Suppose that the eigenvalues of matrix A are α_j then $\alpha_1=-a_1+ib_1$, $\alpha_2=-a_1-ib_1$, $\alpha_3=-a_2+ib_2,\ldots,\alpha_{11}=-a_6+ib_6,\alpha_{12}=-a_6-ib_6$. Therefore, the characteristic equation corresponding to system (8) is the following

$$\prod_{j=1}^{12}\left(\lambda-\alpha_j-\beta_je^{-\lambda\tau}\right)=0 \tag{11}$$

Noting that $Re(\alpha_j)=-a_j<0$, and there exists some $\beta_k>0$ or $Re(\beta_k)>\alpha_k$ thus system (11) has a positive real eigenvalue or an eigenvalue which has a positive real part. Therefore, the trivial solution of system (8) (or (7)) is unstable according to the basic result of delayed differential equation, implying that the trivial solution of system (6) is unstable. Since system (6) has a unique equilibrium point and all solutions are bounded, based on the extended Chafee's criterion [21, 22], this instability of the trivial solution will force system (6) to generate a limit cycle, namely, a periodic oscillatory solution.

Now set $\sigma=\max_{1\le j\le 6}\{-a_j+|-b_j|\}$, $b=\max_{1\le j\le 12}\sum_{i=1}^{12}|b_{ij}|$.

Then we have

Theorem 2 Assume that Lemma 1 and Lemma 2 hold for selecting parameter values of $a_j,\ b_j,\ m_{kj},\ n_{kj}(k,\ j=1,\ 2,\ \ldots,\ 6)$. If

$$\sigma+b>0 \tag{12}$$

Then the unique equilibrium point of system (6) is unstable, implying that system (6) generates a periodic oscillatory solution.

Proof Similar to Theorem 1, we show that the trivial solution of system (7) is unstable, then the trivial solution of system (6) also is unstable. In system (7), let $v(t)=\sum_{j=1}^{6}(|x_j(t)|+|y_j(t)|)$, then we have

$$\frac{dv(t)}{dt} \leq \sigma v(t) + bv(t-\tau) \quad (13)$$

Corresponding to equation (13), we consider the following equation

$$\frac{dw(t)}{dt} = \sigma w(t) + bw(t-\tau) \quad (14)$$

The characteristic equation associated with equation (14) is

$$\lambda = \sigma + be^{-\lambda\tau} \quad (15)$$

We claim that there exists a positive root of (15). Let $f(\lambda) = \lambda - \sigma - be^{-\lambda\tau}$. Obviously, $f(\lambda)$ is a continuous function of λ. When $\lambda = 0$, we get $f(0) = -\sigma - b = -(\sigma + b) < 0$ since $\sigma + b > 0$. On the other hand, there exists a suitably large λ say $\lambda_1 > 0$ such that $f(\lambda_1) = \lambda_1 - \sigma - be^{-\lambda_1\tau} > 0$ since $\lim_{\lambda_1 \to \infty} e^{-\lambda_1\tau} = 0$. Based on the Intermediate Value Theorem, there exists a $\lambda_0 \in (0, \lambda_1)$ such that $f(\lambda_0) = \lambda_0 - \sigma - be^{-\lambda_0\tau} = 0$. In other words, λ_0 is a positive characteristic root of equation (15). Therefore, the trivial solution of equation (14) is unstable. Noting that $v(t) \leq w(t)$. So the instability of the trivial solution of (14) implies that the trivial solution of system (7) (thus system (6)) is unstable. This instability of the trivial solution such that system (6) has a limit cycle, namely, a periodic oscillatory solution.

4. Simulation Result

This simulation is based on system (6). We first select the parameters as $a_1=0.45$, $a_2=0.65$, $a_3=0.48$, $a_4=0.35$, $a_5=0.25$, $a_6=0.18$, $b_1=0.24$, $b_2=0.56$, $b_3=0.24$, $b_4=0.32$, $b_5=0.45$, $b_6=0.32$; $m_{14}=0.56$, $m_{15}=0.42$, $n_{14}=-0.12$, $n_{15}=0.32$, $m_{24}=-0.65$, $m_{25}=0.36$, $m_{26}=0.55$, $n_{24}=0.32$, $n_{25}=0.48$, $n_{26}=0.25$ $m_{35}=-0.65$, $m_{36}=0.45$, $n_{35}=0.36$, $n_{36}=0.36$, $m_{41}=0.35$, $m_{42}=0.58$, $n_{41}=0.35$, $n_{42}=-0.65$, $m_{51}=-0.68$, $m_{52}=0.56$, $m_{53}=0.32$, $n_{51}=0.45$, $n_{52}=-0.25$, $n_{53}=0.25$, $m_{62}=0.48$, $m_{63}=0.32$, $n_{62}=0.12$, $n_{63}=-0.18$. The activation functions $f_{kj}(x_j, y_j) = (\tanh(x_j) + \tanh(y_j)) + i(\tanh(x_j) + \tanh(y_j))$, *thus* $f^R_{kj}(x_j, y_j) = f^I_{kj}(x_j, y_j) = \tanh(x_j) + \tanh(y_j)$, $\frac{\partial f^R_{kj}(0,0)}{\partial x_j} = \frac{\partial f^R_{kj}(0,0)}{\partial y_j} = 1$, and $\frac{\partial f^I_{kj}(0,0)}{\partial x_j} = \frac{\partial f^I_{kj}(0,0)}{\partial y_j} = 1$ $(k, j=1,2,...,6)$, time delay is 0.5. We see that the eigenvalues of matrix B are 0.8903, -0.8903, 0.7645±0.6458 *i*, -07645±0.6558 *i*, 0, 0, 0, 0, 0, 0. Noting that there exists a positive eigenvalue $0.8903 > a_j$. The conditions of Theorem 1 are satisfied. Based on Theorem 1, there exists a periodic oscillatory solution (see Figure 1). In order to see the effect of the time delay, we change time delay as 1.5, the other parameters are the same as in Figure 1, we see that the oscillatory frequency and oscillatory amplitude both are changed (see Figure 2). Then we change the activation function as $f_{kj}(x_j, y_j) = (\arctan(x_j) + \arctan(y_j)) + i(\arctan(x_j) + \arctan(y_j))$, thus we still have $f^R_{kj}(x_j, y_j) = f^I_{kj}(x_j, y_j) = \arctan(x_j) + \arctan(y_j)$ $\frac{\partial f^R_{kj}(0,0)}{\partial x_j} = \frac{\partial f^R_{kj}(0,0)}{\partial y_j} = 1$ and $\frac{\partial f^I_{kj}(0,0)}{\partial x_j} = \frac{\partial f^I_{kj}(0,0)}{\partial y_j} = 1$ $(k, j=1,2,...,6)$, the parameters are the same as in Figure 2, we see that the oscillatory frequency and oscillatory amplitude both are changed slightly (see Figure 3). This means that the activation functions effect the oscillatory behavior not too much. Now we select another set of parameters as $a_1=0.45$, $a_2=1.15$, $a_3=1.28$, $a_4=0.42$, $a_5=0.76$, $a_6=1.35$, $b_1=1.68$, $b_2=0.65$, $b_3=0.92$, $b_4=0.58$, $b_5=0.75$, $b_6=0.85$, $m_{14}=1.25$, $m_{15}=0.76$, $n_{14}=-1.2$, $n_{15}=0.45$, $m_{24}=-0.25$, $m_{25}=0.56$, $m_{26}=0.15$, $n_{24}=0.38$, $n_{25}=1.78$, $n_{26}=0.25$, $m_{35}=-1.95$, $m_{36}=0.45$, $n_{35}=1.36$, $n_{36}=0.24$, $m_{41}=0.65$, $m_{42}=0.98$, $n_{41}=0.45$, $n_{42}=-0.65$, $m_{51}=-0.78$, $m_{52}=0.65$, $m_{53}=0.85$, $n_{51}=0.45$, $n_{52}=-1.15$, $n_{53}=0.36$, $m_{62}=1.28$, $m_{63}=0.32$, $n_{62}=1.12$, $n_{63}=-0.18$. The activation function is as in Figure 3, time delay is 0.6. We see that $\sigma = -0.5$, $b = 7.44$ .Therefore, $\sigma + b > 0$ holds. Based on Theorem 2, there exists a periodic oscillatory solution (see Figure 4).

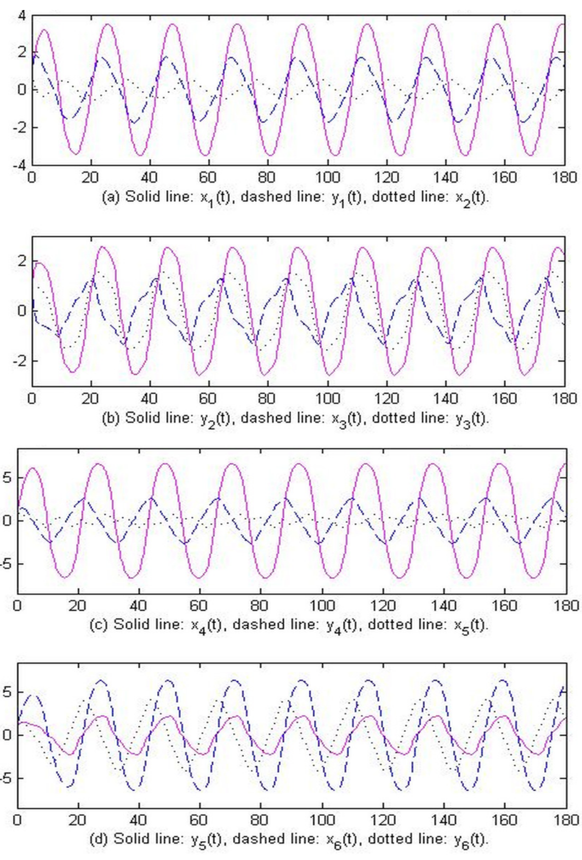

Figure 1. Oscillation of the solutions, activation function: tanh (z), time delay: 0.5.

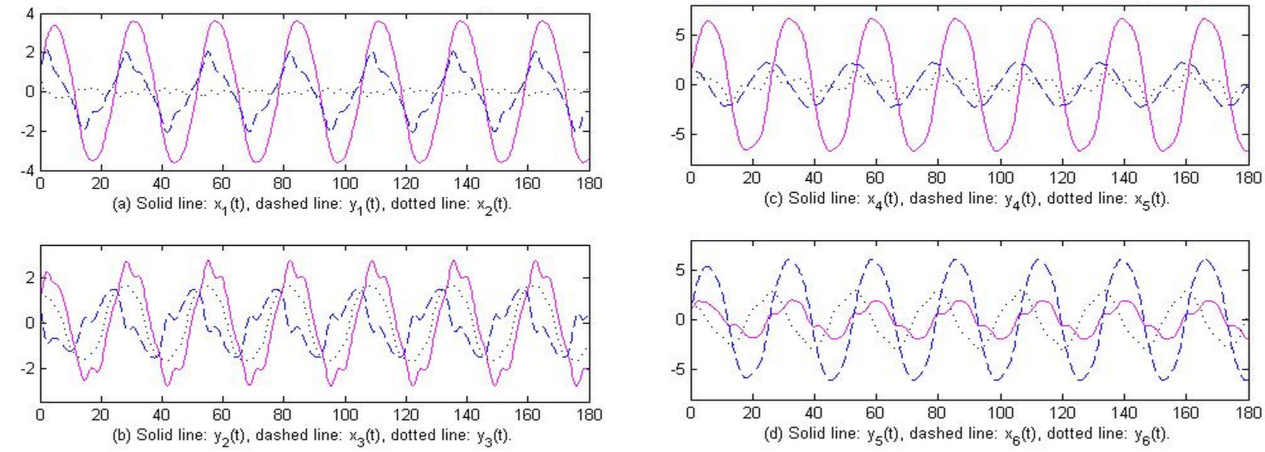

Figure 2. Oscillation of the solutions, activation function: tanh (z), time delay: 1.5.

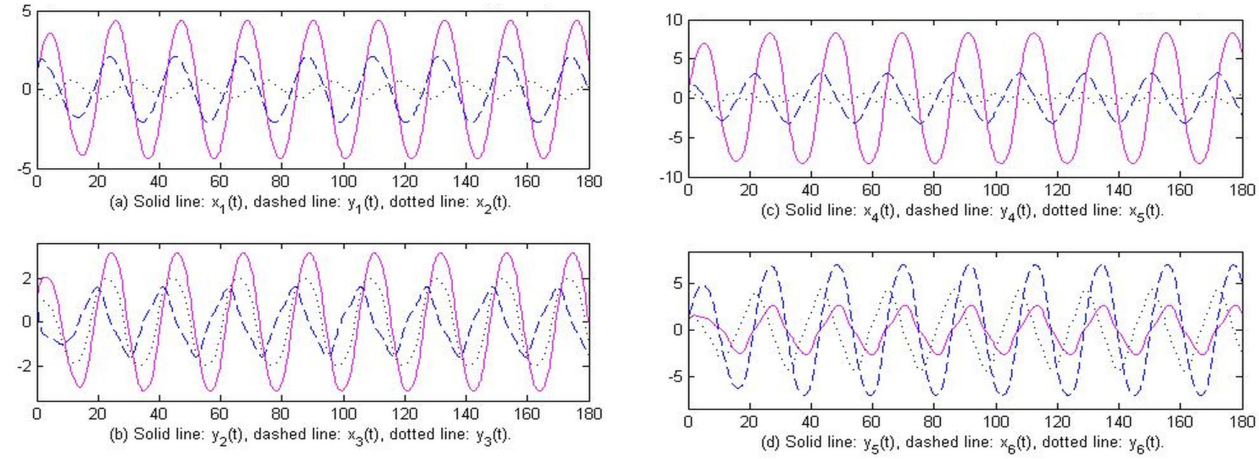

Figure 3. Oscillation of the solutions, activation function: arctan (z), time delay: 0.5.

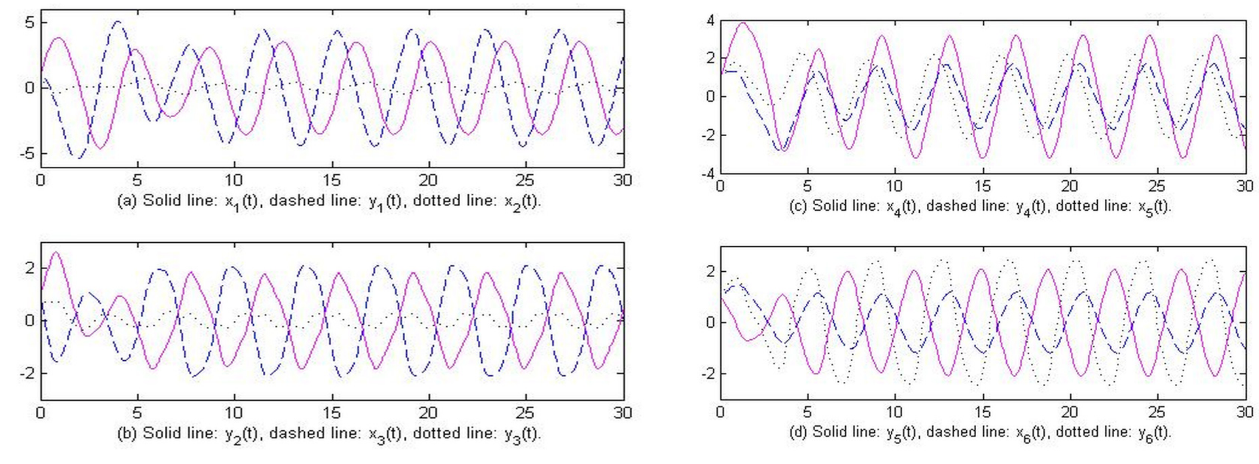

Figure 4. Oscillation of the solutions, activation function: arctan (z), time delay: 0.6.

5. Conclusions

The paper has discussed the oscillatory behavior of the solutions for a complex-valued neural network model with discrete delay. By means of the mathematical analysis method, two criteria to guarantee the existence of periodic oscillatory solution are provided which are easy to be checked. In this network, we decomposed the activation functions and connection weights into their real and imaginary parts, so as to discuss an equivalent real-valued system. The activation functions affect the oscillatory behavior slightly.

Conflict of Interest

The author declares no conflict of interest.

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