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Fractal Geometry: Axioms, Fractal Derivative and Its Geometrical Meaning

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ABSTRACT

Physics success is largely determined by using mathematics. Physics often themselves create the necessary mathematical apparatus. This article shows how you can construct a fractal calculus - mathematics of fractal geometry. In modern scientific literature often write from a firm that "there is no strict definition of fractals", to the more moderate that "objects in a certain sense, fractal and similar." We show that fractal geometry is a strict mathematical theory, defined by their axioms. This methodology allows the geometry of axiomatised naturally define fractal integrals and differentials. Consistent application on your input below the axiom gives the opportunity to develop effective methods of measurement of fractal dimension, geometrical interpretation of fractal derivative gain and open dual symmetry.

1. Axioms of Fractal Geometry

As is often the case in science, after a long preparatory period one person, Benoit Mandelbrot, about 50 years ago opened a new world - the world of fractal geometry^[1-5]. Most natural scope new geometry is physics. Fractal geometry application, however, is lagging behind demand. Description of the nature and variety of its manifestations requires the corresponding mathematical apparatus. Without this, often it is not possible to formulate the initial concepts. Filling this gap with regard to fractal geometry and dedicated to this article.

More than 20 centuries ago Euclid built a system of axioms known at that time, geometry, indicating thereby how science should be built as such. It was realized that the fairness of the axioms may be proved only by comparison with experience – obtaining known results and the

opening of new patterns that are validated experimentally. Axioms of Euclidean geometry by defining the start point, as an object, it's easier that there is nothing (no taste, no color). Physics point gave mass, charge and spin (so you can try it to taste and see its beauty). Fractal geometry begins with next in complexity from the object point - line. Development of science expands the concept of actually existing. In the real world there is no smooth Euclidean lines they are crooked, cut and curves. These lines have a remarkable property of similarity. If drawn square watch from afar, it will have a blurred point. Near see just any side. But any increase is not looked at fractal line, then it will be all the same rugged and curve.

The main property lines that it has length, which can be measured. Putting, for example, metre line to some straight six times, we learn that the length of this line is

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6 meters. In Euclidean geometry, the length of the line is not dependent on the rulers. If you use our direct centimeter scale, then its length does not change, although it is equal to 600 cm. For fractal lines, after numerous measurements, it was found that their length L depends on the scale of measurement χ , and this dependence is exponential. Such power dependence of the record in the form of law the Mandelbrot set:

$$L = C \chi^{1-D} \quad (1)$$

The genius of the Mandelbrot set is reflected in guess, that describes D the new dimension of fractal geometry. A new geometry value C called uncertain factor. Usually $D > 1$, so when $\chi \rightarrow 0$ length $L \rightarrow \infty$. Reducing the scale of measurement, we increase the length of the curve. You can clearly tell that when driving on rough terrain steps should be done as widely as possible. With long legs Wayfarer road will seem shorter. If you're down, the path length to increase sharply. Historical justice demands say that formula (1), borders, first installed the English scientist Richardson about 100 years ago. So sometimes say about law - Mandelbrot of Richardson.

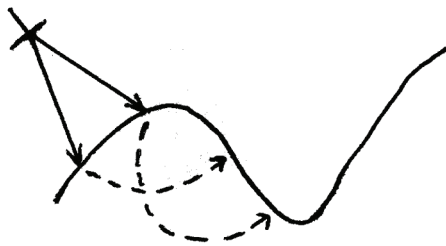


Figure 1. Length measurement compass.

Draw a curve sinus and using a compass with different solutions (scale), measure its length (fig. 1). All measured values "scale χ - length L " on the graph with axes $\ln \chi$ and $\ln L$ will direct. So you can ensure equity formula Mandelbrot-Richardson. Measuring progress in detail given in^[6]. The drawn curve in Fig. 1 try to take into account the similarity property, although it is difficult to demonstrate "by hand".

An important property of fractal curves, their likeness must be given mathematical form. This form you can give only a visceral, obvious way. Then stretched or going to compress the line curve in η again, so the new length will $L^* = \eta L$. The amount of η called a factor. Since similarity means that any portion of the curve is similar to the line, then measuring the new length scale can be done, in η time is different from the original scale, that is $\chi^* = \eta \chi$. Specifically, fractal, izlomannaja and/or indented line, under magnification η times will have long ηL . You can measure the magnitude $\eta \chi$. Therefore, the following formula is valid conditions of similarity:

$$\eta L = C(\eta \chi)^{1-D} \quad (2)$$

Formula (2) means that any plot a fractal line has the same fractal dimensionality. Here brackets are operator-special mathematical validity, meaning that you must first undertake scaling, and only then result in degree. Otherwise, say, the parentheses can be opened only after you specify a scale multiplier η .

Formula Mandelbrot -Richardson and similarity condition have linked two of the three units-the length and scale of the D fractal dimensionality of leaving arbitrary. If the Euclidean geometry of space dimension is given once and for all, the fractal geometry of its dimension for each object is determined only by experience, or, as a rare exception, is calculated.

Euclidean geometry by David Hilbert is 20 axioms. Fractal geometry is based on two additional axioms, mathematical formulation which provides a formula Mandelbrot -Richardson (1) and the condition of similarity (2). It is impossible to prove through logical axioms, they cannot justify themselves, their only practice fairness. Euclid in his book "Beginning" as an application of the axioms of geometry considered 500 tasks. Use only the fact of similarity allows physics associate geometrical and physical values sedate way. Using thermodynamics or quantum field theory, many power-related indicators what is either a physical process, able to express through two randomly selected. One of the main tasks of fractal geometry, it is mixing power indicators for the largest single independent -fractal dimension. In our statement does not dwell on how of the axioms (1) and (2) you can get all recently known results related to sedate indicators^[6]. This can be done as a mathematical exercise. However, let's consider a simple ratio required in our statement.

In space you can always select an arbitrary region linear dimension R , inside will be a fractal curve. With the increase of a linear dimension will increase and the length of the inside area of the fractal lines. To find the length dependence of linear size, enough in the axiom of self-specify a scale multiplier η . Putting $\eta = 1 / R$, after cuts from the formula (2) have a simple result:

$$L \sim R^D \quad (3)$$

Not to be distracted by an unspecified multiplier C with the typed character of proportionality.

2. Measurement of Fractal Dimension

Direct and most robust way to measure the fractal dimension is to use formula Mandelbrot-Richardson. For example, presented at fig. 2 and 3 topographic maps of the Selenga River and Volga river deltas, and rice. Lightning 4, zoom must be chosen to bypass all visible sleeves branches. Less time-consuming method of measurement is to use

formula (3). When the measuring scale is fixed and varies the size of the area. So, in Figure 2. 5 for strimer channels area selected as a circle.

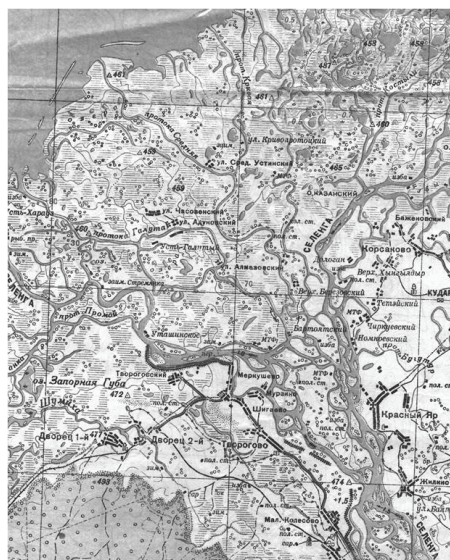


Figure 2. Topographic map of the Selenga Delta

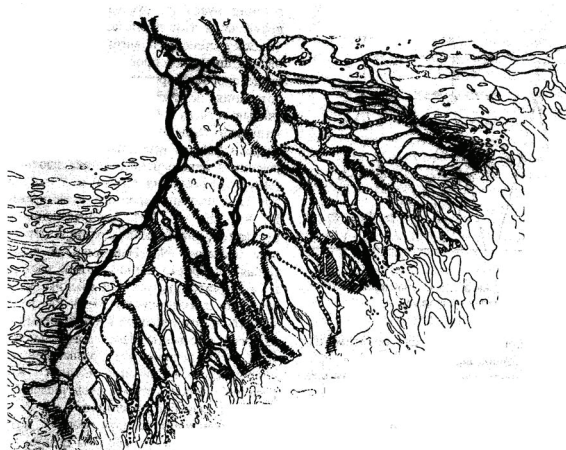


Figure 3. Topographic map of the Delta of the Volga River

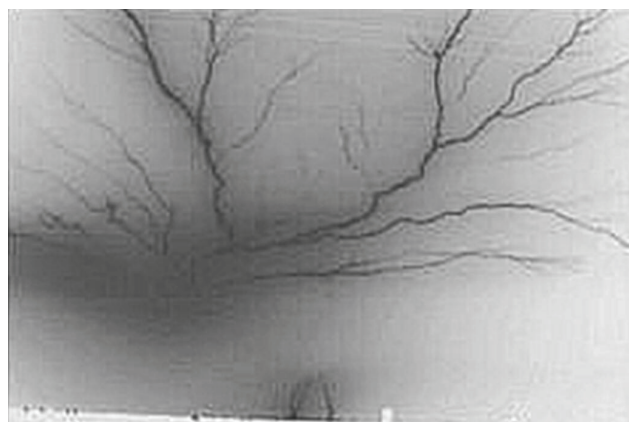


Figure 4. Electric discharges branchy lightning

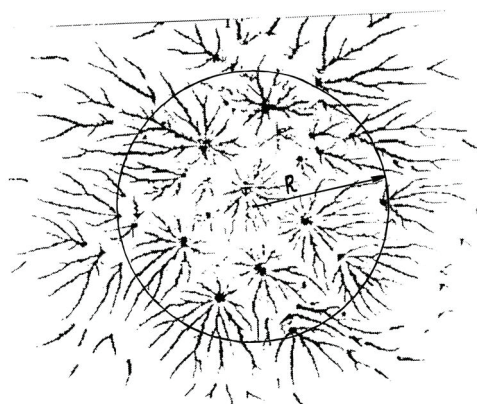


Figure 5. Electric discharges strimer channels

Presented in Figure 2-5 natural formation can be attributed to a class of branched structures. They have a distinctive feature-from some Center there are a growing number of branches, branches. With the number of branches N sedate manner depends on the size of the area R , i.e.

$$R \sim N^{1/\nu} \quad (4)$$

Power indicator ν called dimension of wandering. This name stems from the fact that historically the first law (4) indicator $\nu = 2$ в 1905 г. got a. Einstein to Brownian motion. Then, in 1938 year Kolmogorov turbulence for advanced fixed $\nu = 2/3$, and in 1948, P. Flory found $\nu = 5/3$ for polymer chains. While wandering along the particles, for example, fractal dimension equal to wandering line fractal dimension. Minimizing a polymer chain in a tangle on the plane $\nu_2 = 1 + D_2/2$. Index 2 indicates that values are defined for flat projection. Fractal derivative, as seen, for Planar Projection branched structures establishes the following communication dimension of wandering with fractal dimension:

$$\nu_2 = 2(D_2 - 1) \quad (5)$$

(4) and (5) allow you to develop the most simple method of measurement of fractal dimension. For this purpose it is enough to only count the number of intersections of the selection border with forks. For example, in Figure 5 the number $N = 53$. Changing linear dimensions, every time we find a new value for the number of intersections. The graph with axes $\ln R$ and $\ln N$ all the measured number will direct. For each object in Figures 2-5 used all three of these measuring methods that gave consistent results. Thus, it is established that

$$D(\text{Selenga River}) = 1.38 \pm 0.02; D(\text{Volga River}) = 1.72 \pm 0.02; \\ D(\text{strimer}) = 1.53 \pm 0.02; D(\text{Lightning}) = 1.73 \pm 0.04.$$

Fractal line, located on the plane can have dimension only smaller 2. In space its dimension may take the maximum value equal to 3. Presented in Figure 2 branched structures are planning projections, so the dimensionality of less 2. Fractal dimension of objects on a plane above

identified as D_2 . For spatial objects naturally enter fractal dimension D_3 .

3. Fractal Calculus

Begin to build the mathematical apparatus of fractal geometry - is fractal calculation. This building can hold is similar to traditional integro-differential calculus. Length fractal lines, by definition, is the sum of all sizes: $L = \sum \chi$. Similarly, the traditional definition of integral, sum $\sum \chi$ replace the following integral, which call fractal:

$$\sum \chi = \int \chi d_D \chi \quad (6)$$

Note that the icon D , fractal, pointing at the bottom write differential d . Since the length of the line is $C\chi^{1-D}$, then come to the first rule of fractal calculus – rule integration of linear functions:

$$\int \chi d_D \chi = C \chi^{1-D} \quad (7)$$

Having the formula (7) large-scale conversion $\chi \rightarrow \eta \chi$, you can get the law of similarity to fractal differential:

$$d_D \eta \chi = \eta^{-D} d_D \chi \quad (8)$$

In this expression, clearly visible difference fractal differential of fractional order differentials, for the latest $d^D \eta \chi = \eta^D d^D \chi$ [7].

Calculation of square S surface fractal is similar to equation (1), only with the obvious generalization:

$$S = C \chi^{2-D} \quad (9)$$

Similarity condition will have the following form:

$$\eta S = C (\eta \chi)^{2-D} \quad (10)$$

When $D = 2$ have a "true" smooth surface. Surface with fractal dimension $D = 3$ so its hinges and cut that densely fills the entire volume.

Cut squares of size fractal $\chi \times \chi$. Its total area will be $\sum \chi^2$. Replacing the sum by an integral fractal, and bearing in mind (9), we have a rule of fractal integration of quadratic functions:

$$\int \chi^2 d_D \chi = C \chi^{2-D} \quad (11)$$

Generalization to an arbitrary power function obviously leads to the following rule:

$$\int \chi^n d_D \chi = C \chi^{n-D} \quad (12)$$

Here n is arbitrary, can and fractional number.

As with the traditional calculation, we assume that fractal derivative - is inverse to the fractal integration operation. Thus, we believe that

$$\left(\frac{d}{d\chi} \right)_D \int \varphi(\chi) d_D \chi = \varphi(\chi) \quad (13)$$

It is now easy to install fractal rules of derivation of elementary functions. Dropping a simple calculation, write an expression for the fractal derivative power function:

$$\left(\frac{d}{d\chi} \right)_D \chi^n = C^{-1} \chi^{n+D} \quad (14)$$

Laws of motion of material bodies are specified in the form of differential equations. It can be assumed that the movement in fractal medium should be described in the form of some fractal-differential equations. As such, might find application in a subsequent, we present below a simple equations and their solutions:

$$\left(\frac{d}{dx} \right)_D y = A, \quad y = A C x^{-D};$$

$$\left(\frac{d}{dx} \right)_D y = x^n, \quad y = C x^{n-D};$$

$$\left(\frac{d}{dx} \right)_D y + k^{-D} y = 0, \quad y = \exp(k x (-1)^{1/D})$$

4. The Geometric Meaning of Fractal Derivative

One of the properties of fractal lines, their broken at each point. This means that these curves are not ordinary derivatives. If you draw a checkmark, it will have two break point tangents. And because the tangent and geometric interpretation is derived, it turns out that check mark in point break has two derivatives. In essence, the checkmark, there is one point in the place where it is unknown what it is necessary to take the derivative. It is clear that fractal objects have irregular at each point, it is necessary to introduce a special – fractal derivative, and so it is a natural generalization of the normal derivative as it is given by the formula (13).

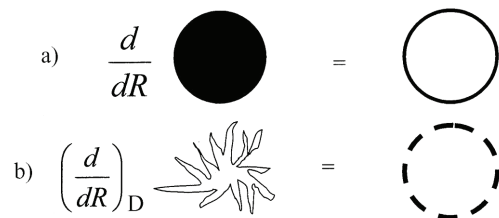


Figure 6. The geometric meaning of ordinary a) and fractal b) derivatives

Consider a circle of radius R . Taking the derivative of the area of the circle, we get the circumference (Figure 6a). See that actually cut out the inner part, leaving only the border of the circle. Similarly, we assume that the effect of fractal derivative comes to removal areas did not come into contact with a circle radius R (Figure 6b).

The length of all branches within the linear dimension R is proportional to R^D . Taking fractal derivative, we cut out everything, leaving only the ground border area with branches. Dividing the expression obtained in the area get the number of intersections. Using the rule of taking fractal derivative of power function (14), the number of branches of planar projection will have

$$N = \frac{1}{R^2} \left(\frac{d}{dR} \right)^D R^D = C \cdot R^{2D-2} \quad (15)$$

Comparing the expression obtained with (4), come to the result (5).

In the three-dimensional case, to get the number of branches, fractal derives from lengths of all branches must be divided by volume R^3 , i.e. for spatial structure:

$$N = \frac{1}{R^3} \left(\frac{d}{dR} \right)^D R^D = C R^{2D-3} \quad (16)$$

It follows that for a spatial branching

$$\nu_3 = 2D_3 - 3 \quad (17)$$

For a variety of natural fractal objects must be installed separately, the relationship between D_2 and D_3 their relationship with dimensions respectively. Formula (5) and (17) give an example of connection ν and D for branched structures. Still, you need to install a dependency D_2 and D_3 among themselves.

5. The dual symmetry

For polymer chains $\nu = 1 + D/2$, for branched structures $\nu = 2(D-1)$. If you replace $\nu \leftrightarrow D$, both expressions will go at each other. It turns out that polymer chains and branched structures have a remarkable symmetry-they are duals of each other. Flory thermodynamic method found that polymer chains for their fractal dimension easily expressed through the dimension of E Euclidean space, where chains are attached [8]:

$$D_E = \frac{E+2}{3}.$$

Отсюда следует, что $D_3 = D_2 + \frac{1}{3}$. According to the dual symmetry, replacing D on ν , we find that for the branched structures must be $\nu_3 = \nu_2 + \frac{1}{3}$. Using (5) and

(17), first get $2D_3 - 3 = 2D_2 - 2 + \frac{1}{3}$, from where

$$D_3 = D_2 + \frac{2}{3} \quad (18)$$

This is the original ratio of the connection D_2 and D_3 for branched structures.

Thermodynamic method for the spatial education strimer channels, you can install that $N(R) \sim R^{7/5}$. This means that $\nu_3 = 7/5$. Then from (17) first find $D_3 = 11/5$, then

from (5) permanently $D_2 = 23/15 = 1.53$, that is consistent with the above measured value of fractal dimension of Planar Projection strimer channels.

6. Conclusion

Article pictures and formulas should help you understand and appreciate the beauty and complexity of fractal geometry. Geometric interpretation of fractal derivative shown is not the pinnacle of fractal calculus. Rather, we are at the beginning of the road. To date received values of fractal integrals and differentials for some elementary functions, but many features are still awaiting consideration. Fractal dimension, as seen, takes a non-integral values, so we could talk about fractional-dimensional values. However, this was not done deliberately. After opening Mandelbrot new geometry of its mathematical apparatus was directly identified with fractional-dimensional integrals and differentials. But this identification is incorrect, it can be seen from the fact that, in fractal geometry inherent is the presence of undetermined multiplier. And in the fractional-dimension calculus all values defined.

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