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ARTICLE

# New Approach to Observer-Based Finite-Time $H_\infty$ Control of Discrete-Time One-Sided Lipschitz Systems with Uncertainties

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ABSTRACT

This paper investigates the finite-time $H_\infty$ control problem for a class of nonlinear discrete-time one-sided Lipschitz systems with uncertainties. Using the one-sided Lipschitz and quadratically inner-bounded conditions, the authors derive less conservative criterion for the controller design and observer design. A new criterion is proposed to ensure the closed-loop system is finite-time bounded (FTB). The sufficient conditions are established to ensure the closed-loop system is $H_\infty$ finite-time bounded ($H_\infty$ FTB) in terms of matrix inequalities. The controller gains and observer gains are given. A numerical example is provided to demonstrate the effectiveness of the proposed results.

## 1. Introduction

It is widely accepted that a large percentage of systems are nonlinear in nature. As a result, many studies on nonlinear systems have been conducted in the previous several decades. However, most of the times, nonlinearities discussed in these papers focus on traditional Lipschitz condition [1-4]. It is worth noting that the Lipschitz nonlinear system in the above literature is usually only applicable to some nonlinear systems with sufficiently small Lipschitz constant. The so-called one-sided Lipschitz nonlinear system was developed to overcome this difficulty. Later, quadratically inner-bounded condition was proposed by Abbaszadeh and Marquez [5]. It is worth noting that the traditional Lipschitz system is a special case of one-sided Lipschitz system and quadratic inner bounded system. Therefore, nonlinear systems satisfying quadratic inner boundedness condition and one-sided Lipschitz condition describe a wider class of nonlinear systems.

In practical engineering, many control problems can

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be summed up as $H_\infty$ standard control problems: interference suppression problem, tracking problem, and robust stability problem. Because of its practical importance, the $H_\infty$ control problem has always been an important research topic. The main purpose of $H_\infty$ control is based on reducing the influence of external disturbance input on the adjustable output of the system. In order to ensure the required robust stability, some results have been obtained on the design of $H_\infty$ control [6-11]. This control is usually available on the assumption that the entire state is accessible. However, in numerous cases, this assumption is invalid, so it is very important to construct an observer that can provide the estimated value of the system state [12]. In recent years, the observer-based control has attracted the attention of researchers, and some results have been obtained [13,14].

Moreover, in some practical cases, the system state cannot exceed a defined boundary within a finite time interval. Hence, the finite-time transient performance should be considered. Recently, finite-time stability and $H_\infty$ control problems have gradually become a well-researched topic and have been applied to many systems [15-18]. In 2020, Wang J X et al. [15] considered the problem of robust finite-time stabilization for uncertain discrete-time linear singular systems. Feng T et al. [16] studied the problem of finite time stability and stabilization for fractional-order switched singular continuous-time system. Zhang T L et al. [17] looked at the finite-time stability and stabilization for linear discrete stochastic systems.

However, so far, the problem of observer-based finite-time $H_\infty$ control for discrete-time one-sided Lipschitz systems have not been fully addressed, which leads to the main purpose of our research.

In this paper, the observer-based finite-time $H_\infty$ controller for nonlinear discrete-time system with uncertainties is studied. We design the observer and observer-based controller. Using Lyapunov function approach and some lemmas, we obtain the criterion of $H_\infty$ FTB for the closed-loop system. Finally, the validity of the proposed method is demonstrated by a numerical example.

This paper is organized as follows. Section 2 covers some preliminary information as well as the problem statement. In Section 3, the sufficient conditions of FTB an $H_\infty$ FTB for nonlinear discrete-time systems are established. In Section 4, a numerical example is presented. Conclusions are given in Section 5.

**Notations** $R^n$ denotes the $n$-dimensional Euclidean space. $*$ denotes a block of symmetry. $B<0 (B>0)$ denotes the matrix $B$ is a negative definite (positive definite) symmetric matrix. We define $He(S)=S+S^T$. $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denotes the maximum eigenvalue and minimum eigenvalue of a matrix respectively. $\langle , \rangle$ is inner product in the space $R^n$, i.e. given $x, y\in R^n$, then $\langle x,y\rangle = x^T y$. $\mathbb{N}$ denotes the non-negative integer set.

## 2. Problem Formulation

Consider the following uncertain one-sided Lipschitz discrete-time system:

$$\begin{cases} x_{k+1} = (A+\Delta A(k))x_k + g(x_k) + Bu_k + w_k, \\ y_k = (C+\Delta C(k))x_k, \\ z_k = Ex_k + Fw_k, \end{cases} \tag{1}$$

where $x_k \in R^n$ is the $n$-dimensional state vector, $y_k\in R^l$ is the output measurement, and $u_k\in R^m$ is the control input. $z_k\in R^q$ is the control output. The disturbance $w_k \in R^p$ satisfies:

$$\sum_{k=0}^{N} w_k^T w_k \le d^2, d\ge 0. \tag{2}$$

$A, B, C, E$ and $F$ are known real constant matrices. $\Delta A(k)$ and $\Delta C(k)$ are time-varying matrices, which are assumed to be of the form:

$$\Delta A(k) = M_1\Delta_1(k)N_1, \quad \Delta C(k) = M_2\Delta_2(k)N_2, \tag{3}$$

where $M_1, M_2, N_1$ and $N_2$ are known real constant matrices, and $\Delta_i(k) (i=1,2)$ are the unknown time-varying matrix-valued function subject to the following conditions:

$$\Delta_i^T(k)\Delta_i(k) \le I, \quad \forall k\in\mathbb{N}, \quad i=1,2. \tag{4}$$

$g(x_k)$ is a nonlinear function satisfying the following assumptions.

**Assumption 1** [19] $g(x)$ verifies the one-sided Lipschitz condition:

$$\langle g(x)-g(\hat{x}), x-\hat{x}\rangle \le \rho\|x-\hat{x}\|^2, \quad \forall x,\hat{x}\in R^n, \tag{5}$$

where $\rho$ is the so-called one sided Lipschitz constant.

**Assumption 2** [19] $g(x)$ verifies the quadratic inner-bounded condition:

$$\|g(x)-g(\hat{x})\|^2 \le \beta\|x-\hat{x}\|^2 + \alpha\langle g(x)-g(\hat{x}), x-\hat{x}\rangle, \quad \forall x,\hat{x}\in R^n, \tag{6}$$

where $\alpha$ and $\beta$ are known constants.

**Remark 1** Different from the traditional Lipschitz condition, constant $\rho$, $\alpha$ and $\beta$ in the nonlinearity considered here can be negative, positive or zero.

In this paper, we construct the following state observer-based controller:

$$\begin{cases} \hat{x}_{k+1} = A\hat{x}_k + g(\hat{x}_k) + Bu_k + L(y_k - C\hat{x}_k), \\ u_k = -K\hat{x}_k, \end{cases} \tag{7}$$

where $\hat{x}_k$ is the estimate of $x_k$, $K$ and $L$ are the controller and observer gains, respectively, to be designed.

Let $e_k = x_k - \hat{x}_k$. Then we have:

$$\begin{aligned} e_{k+1} &= x_{k+1} - \hat{x}_{k+1} \\ &= (A-LC)e_k + (\Delta A - L\Delta C)x_k + \tilde{g}(x_k,\hat{x}_k) + w_k, \end{aligned}$$

where $\tilde{g}(x_k,\hat{x}_k)=g(x_k)-g(\hat{x}_k)$.

Let $\bar{x}_k=\begin{bmatrix} x_k^T & e_k^T \end{bmatrix}^T$. The closed-loop system can be written as:

$$\bar{x}_{k+1}=\bar{A}\bar{x}_k+\bar{g}(\bar{x}_k)+\bar{I}w_k, \tag{8}$$

where,

$$\bar{A}=\begin{bmatrix} A+\Delta A-BK & BK \\ \Delta A-L\Delta C & A-LC \end{bmatrix}, \quad \bar{g}(\bar{x}_k)=\begin{bmatrix} g(x_k) \\ \tilde{g}(x_k,\hat{x}_k) \end{bmatrix}, \quad \bar{I}=\begin{bmatrix} I \\ I \end{bmatrix}. \tag{9}$$

The following definitions and some useful lemmas are introduced to establish our main results.

**Definition 1** (FTB) The closed-loop system (8) is said to be FTB with respect to $(c_1,c_2,N,\bar{R})$, where $0<c_1<c_2$, $\bar{R}>0$, if

$$\bar{x}_0^T\bar{R}\bar{x}_0\le c_1\Rightarrow \bar{x}_k^T\bar{R}\bar{x}_k<c_2, \forall k\in\{1,2,\cdots,N\}. \tag{10}$$

**Definition 2** ($H_\infty$ FTB) The closed-loop system (8) is said to be $H_\infty$ FTB with respect to $(c_1,c_2,N,\bar{R},\gamma)$, where $0<c_1<c_2$, $\bar{R}>0$, if the system (8) is FTB with respect to $(c_1,c_2,N,\bar{R})$ and under the zero-initial condition the following condition is satisfied

$$\sum_{k=0}^{N} z_k^T z_k<\gamma^2\sum_{k=0}^{N} w_k^T w_k, \tag{11}$$

where $\gamma$ is a prescribed positive scalar.

**Lemma 1** [18] Given constant matrices $X_1, X_2$ and $X_3$, where $X_1=X_1^T$, $X_2=X_2^T>0$, then we obtain that $X_1+X_3^T X_2^{-1} X_3<0$ if and only if $\begin{bmatrix} X_1 & X_3^T \\ X_3 & -X_2 \end{bmatrix}<0$.

**Lemma 2** [10] Let $D,S$ and $\Delta$ be real matrices with appropriate dimensions and $\Delta^T\Delta\le I$, the following inequality holds:

$$D\Delta S+S^T\Delta^T D^T\le\frac{1}{\eta}DD^T+\eta S^T S. \tag{12}$$

**Lemma 3** [11] For matrices $\Lambda_1,\Lambda_2,\Lambda_3$ and $\Phi$ with appropriate dimensions and scalar $\varphi$, the following inequality holds,

$$\Lambda_1+\Lambda_3^T\Lambda_2^T+\Lambda_2\Lambda_3<0,$$

if the following conditions satisfied:

$$\begin{bmatrix} \Lambda_1 & \varphi\Lambda_2+\Lambda_3^T\Phi^T \\ * & -\varphi\Phi-\varphi\Phi^T \end{bmatrix}<0. \tag{13}$$

The goal of this paper is to construct an observer-based controller such that the system (8) is $H_\infty$ FTB.

## 3. Main Results

In this part, sufficient conditions for FTB and $H_\infty$ FTB of the system (8) via observer-based controller are developed.

### 3.1 Finite-time Boundedness

For expression convenience, we denote,

$$\theta_1=\rho\varepsilon_1+\beta\varepsilon_2, \theta_2=-\frac{1}{2}\varepsilon_1 I+\frac{1}{2}\alpha\varepsilon_2 I, \quad \theta_3=\rho\varepsilon_3+\beta\varepsilon_4,$$

$$\theta_4=-\frac{1}{2}\varepsilon_3 I+\frac{1}{2}\alpha\varepsilon_4 I, \quad \bar{P}=\begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}, \quad \bar{R}=\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix},$$

$$\lambda_1=\lambda_{\max}(\bar{R}^{-\frac{1}{2}}\bar{P}\bar{R}^{-\frac{1}{2}}), \lambda_2=\lambda_{\min}(\bar{R}^{-\frac{1}{2}}\bar{P}\bar{R}^{-\frac{1}{2}}), \lambda_3=\lambda_{\max}(Q).$$

**Theorem 1** Under Assumptions 1 and 2, system (8) is FTB with respect to $(c_1,c_2,\bar{R},N)$, if there exist a known scalar $\varphi$, scalars $\mu\ge 1$, $\varepsilon_1>0, \varepsilon_2>0, \varepsilon_3>0$, $\varepsilon_4>0$, $\eta_1,\eta_2$, symmetric matrices $P>0, Q>0$, and matrices $Y,V,\Phi$, such that the following inequalities hold:

$$(1+\mu)^N(\lambda_1 c_1+\lambda_3 d^2)<c_2\lambda_2, \tag{14}$$

$$\begin{bmatrix}
\Theta_{11} & 0 & \Theta_{13} & 0 & 0 & \Theta_{16} & 0 & \Theta_{18} & 0 & 0 \\
* & \Theta_{22} & 0 & \Theta_{24} & 0 & \Theta_{26} & \Theta_{27} & \Theta_{28} & 0 & 0 \\
* & * & \Theta_{33} & 0 & 0 & P & 0 & 0 & 0 & 0 \\
* & * & * & \Theta_{44} & 0 & 0 & P & 0 & 0 & 0 \\
* & * & * & * & \Theta_{55} & P & P & 0 & 0 & 0 \\
* & * & * & * & * & \Theta_{66} & 0 & \Theta_{68} & \Theta_{69} & 0 \\
* & * & * & * & * & * & \Theta_{77} & 0 & \Theta_{79} & \Theta_{710} \\
* & * & * & * & * & * & * & \Theta_{88} & 0 & 0 \\
* & * & * & * & * & * & * & * & \Theta_{99} & 0 \\
* & * & * & * & * & * & * & * & * & \Theta_{1010}
\end{bmatrix}<0, \tag{15}$$

where

$$\begin{aligned}
&\Theta_{11}=-(1+\mu)P+\theta_1 I+\eta_1 N_1^T N_1+\eta_2 N_2^T N_2, \\
&\Theta_{13}=\theta_2 I, \quad \Theta_{16}=A^T P-V^T B^T, \quad \Theta_{18}=-V^T, \\
&\Theta_{22}=\theta_3 I-(1+\mu)P, \quad \Theta_{24}=\theta_4 I \quad \Theta_{26}=V^T B^T, \\
&\Theta_{27}=A^T P-C^T Y^T, \quad \Theta_{28}=V^T, \quad \Theta_{33}=-\varepsilon_2 I, \\
&\Theta_{44}=-\varepsilon_4 I, \quad \Theta_{55}=-Q, \quad \Theta_{66}=-P, \\
&\Theta_{68}=\varphi(PB-B\Phi), \quad \Theta_{69}=PM_1, \quad \Theta_{77}=-P, \\
&\Theta_{79}=PM_2, \quad \Theta_{710}=-YM_2, \quad \Theta_{88}=-\varphi\Phi-\varphi\Phi^T, \\
&\Theta_{99}=-\eta_1 I, \quad \Theta_{1010}=-\eta_2 I.
\end{aligned} \tag{16}$$

Furthermore, the controller gain is given by $K=\Phi^{-1}V$ and observer gain is $L=P^{-1}Y$.

**Proof** We first prove that the system (8) is FTB. So, we define the Lyapunov functional candidate as:

$$V_k=x_k^T Px_k+e_k^T Pe_k, \; P>0. \tag{17}$$

Then, we have

$$\begin{aligned}&\Delta V_k-\mu V_k-w_k^TQw_k\\&=x_{k+1}^TPx_{k+1}+e_{k+1}^TPe_{k+1}-(1+\mu)e_k^TPe_k-(1+\mu)x_k^TPx_k\\&\quad-w_k^TQw_k\\&=[(A+\Delta A-BK)x_k+BKe_k+g(x_k)+w_k]^T\\&\quad\times P[(A+\Delta A-BK)x_k+BKe_k+g(x_k)+w_k]\\&\quad+[(A-LC)e_k+(\Delta A-L\Delta C)x_k+\tilde{g}(x_k,\hat{x}_k)+w_k]^T\\&\quad\times P[(A-LC)e_k+(\Delta A-L\Delta C)x_k+\tilde{g}(x_k,\hat{x}_k)+w_k]\\&\quad-(1+\mu)e_k^TPe_k-(1+\mu)x_k^TPx_k-w_k^TQw_k,\end{aligned}\tag{18}$$

From (18), we get that

$$\Delta V_k-\mu V_k-w_k^TQw_k=\varsigma_k^T\Pi\varsigma_k,\tag{19}$$

where

$$\varsigma_k=\begin{bmatrix}x_k^T & e_k^T & g^T(x_k) & \tilde{g}^T(x_k,\hat{x}_k) & w_k^T\end{bmatrix}^T,$$

$$\Pi=\begin{bmatrix}\Pi_{11}&\Pi_{12}&\Pi_{13}&\Pi_{14}&\Pi_{15}\\ *&\Pi_{22}&\Pi_{23}&\Pi_{24}&\Pi_{25}\\ *&*&P&0&P\\ *&*&*&P&P\\ *&*&*&*&2P-Q\end{bmatrix},$$

$$\begin{aligned}&\Pi_{11}=(A+\Delta A-BK)^TP(A+\Delta A-BK)\\&\qquad+(\Delta A-L\Delta C)^TP(\Delta A-L\Delta C)-(1+\mu)P,\\&\Pi_{12}=(A+\Delta A-BK)^TPBK+(\Delta A-L\Delta C)^T\times\\&\qquad P(A-LC),\\&\Pi_{13}=(A+\Delta A-BK)^TP,\ \Pi_{14}=(\Delta A-L\Delta C)^TP,\\&\Pi_{15}=(A+\Delta A-BK)^TP+(\Delta A-L\Delta C)^TP,\\&\Pi_{22}=(BK)^TPBK+(A-LC)^TP(A-LC)\\&\qquad-(1+\mu)P,\\&\Pi_{23}=(BK)^TP,\quad \Pi_{24}=(A-LC)^TP,\\&\Pi_{25}=(BK)^TP+(A-LC)^TP.\end{aligned}$$

From (5) and (6), it follows that

$$\varepsilon_1\rho x_k^Tx_k-\frac{1}{2}\varepsilon_1g^T(x_k)x_k-\frac{1}{2}\varepsilon_1x_k^Tg(x_k)\ge 0,\tag{20a}$$

$$\begin{aligned}&\varepsilon_2\beta x_k^Tx_k+\frac{1}{2}\varepsilon_2\alpha x_k^Tg(x_k)+\frac{1}{2}\varepsilon_2\alpha g^T(x_k)x_k\\&-\varepsilon_2g^T(x_k)g(x_k)\ge 0,\end{aligned}\tag{20b}$$

$$-\varepsilon_3\rho e_k^Te_k-\frac{1}{2}\varepsilon_3\tilde{g}^T(x_k,\hat{x}_k)e_k-\frac{1}{2}\varepsilon_3e_k^T\tilde{g}(x_k,\hat{x}_k)\ge 0,\tag{20c}$$

$$\begin{aligned}&\varepsilon_4\beta e_k^Te_k+\frac{1}{2}\varepsilon_4\alpha e_k^T\tilde{g}(x_k,\hat{x}_k)+\frac{1}{2}\alpha\varepsilon_4\tilde{g}^T(x_k,\hat{x}_k)e_k\\&-\varepsilon_4\tilde{g}^T(x_k,\hat{x}_k)\tilde{g}(x_k,\hat{x}_k)\ge 0,\end{aligned}\tag{20d}$$

where $\varepsilon_1,\varepsilon_2,\varepsilon_3$ and $\varepsilon_4$ are arbitrary strictly positive scalars.

From the inequality (20a-20d), we can obtain that

$$\varsigma_k^T\begin{bmatrix}\theta_1I&0&\theta_2I&0&0\\ *&\theta_3I&0&\theta_4I&0\\ *&*&-\varepsilon_2I&0&0\\ *&*&*&-\varepsilon_4I&0\\ *&*&*&*&0\end{bmatrix}\varsigma_k\ge 0.\tag{21}$$

Then, the following inequality is obtained by adding the left hand side of (21) to (19)

$$\Delta V_k-\mu V_k-w_k^TQw_k\le\varsigma_k^T\Omega\varsigma_k,$$

where

$$\Omega=\begin{bmatrix}\Omega_{11}&\Pi_{12}&\Omega_{13}&\Pi_{14}&\Pi_{15}\\ *&\Omega_{22}&\Pi_{23}&\Omega_{24}&\Pi_{25}\\ *&*&\Omega_{33}&\Pi_{34}&P\\ *&*&*&\Omega_{44}&P\\ *&*&*&*&2P-Q\end{bmatrix},\tag{22}$$

$$\begin{aligned}&\Omega_{11}=\Pi_{11}+\theta_1I,\quad \Omega_{13}=\Pi_{13}+\theta_2I,\\&\Omega_{22}=\Pi_{22}+\theta_3I,\quad \Omega_{24}=\Pi_{24}+\theta_4I,\\&\Omega_{33}=P-\varepsilon_2I,\quad \Omega_{44}=P-\varepsilon_4I.\end{aligned}$$

By using Lemma 1, it is obvious that $\Omega<0$ is equivalent to

$$\Xi=\begin{bmatrix}\Xi_{11}&0&\theta_2I&0&0&\Xi_{16}&\Xi_{17}\\ *&\Xi_{22}&0&\theta_4I&0&\Xi_{26}&\Xi_{27}\\ *&*&\Xi_{33}&0&0&P&0\\ *&*&*&\Xi_{44}&0&0&P\\ *&*&*&*&-Q&P&P\\ *&*&*&*&*&-P&0\\ *&*&*&*&*&*&-P\end{bmatrix}<0,\tag{23}$$

where

$$\begin{aligned}&\Xi_{11}=-(1+\mu)P+\theta_1I,\quad \Xi_{16}=(A+\Delta A-BK)^TP,\\&\Xi_{17}=(\Delta A-L\Delta C)^TP,\quad \Xi_{22}=\theta_3I-(1+\mu)P,\\&\Xi_{26}=(BK)^TP,\quad \Xi_{27}=(A-LC)^TP,\\&\Xi_{33}=-\varepsilon_2I,\quad \Xi_{44}=-\varepsilon_4I.\end{aligned}$$

By segregating the matrix (23) for known and uncertain parts, yield

$$\Xi = \begin{bmatrix} \Xi_{11} & 0 & \theta_2 I & 0 & 0 & (A-BK)^T P & 0 \\ * & \Xi_{22} & 0 & \theta_4 I & 0 & (BK)^T P & (A-LC)^T P \\ * & * & \Xi_{33} & 0 & 0 & P & 0 \\ * & * & * & \Xi_{44} & 0 & 0 & P \\ * & * & * & * & -Q & P & P \\ * & * & * & * & * & -P & 0 \\ * & * & * & * & * & * & -P \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \Delta A^T P & (\Delta A - L\Delta C)^T P \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 \end{bmatrix} < 0. \tag{24}$$

Then, inequality (24) can be rewritten as:

$$\Xi = \Xi_1 + \Xi_2 + \Xi_3 < 0, \tag{25}$$

where

$$\Xi_1 = \begin{bmatrix} \Xi_{11} & 0 & \theta_2 I & 0 & 0 & A^T P & 0 \\ * & \Xi_{22} & 0 & \theta_4 I & 0 & 0 & (A-LC)^T P \\ * & * & \Xi_{33} & 0 & 0 & P & 0 \\ * & * & * & \Xi_{44} & 0 & 0 & P \\ * & * & * & * & -Q & P & P \\ * & * & * & * & * & -P & 0 \\ * & * & * & * & * & * & -P \end{bmatrix},$$

$$\Xi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \Delta A^T P & (\Delta A - L\Delta C)^T P \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 \end{bmatrix},$$

$$\Xi_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -(BK)^T P & 0 \\ * & 0 & 0 & 0 & 0 & (BK)^T P & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 \end{bmatrix}.$$

In order to deal with the nonlinear term $K^T B^T P$ and $-K^T B^T P$ in (25), a non-singular matrix $\Phi$ is introduced and defined $K = \Phi^{-1} V$, it is obvious that

$$P^T BK = (P^T B - B\Phi)\Phi^{-1} V + BV. \tag{26}$$

From (26), we can deduce

$$\Xi = \overbrace{\tilde{\Xi}_1 + \Xi_2}^{\Lambda_1} + He\left[ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ PB - B\Phi \\ 0 \end{bmatrix}}_{\Lambda_2} \underbrace{\Phi^{-1}\begin{bmatrix} -V & V & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\Lambda_3} \right] < 0, \tag{27}$$

where

$$\tilde{\Xi}_1 = \begin{bmatrix} \Xi_{11} & 0 & \theta_2 I & 0 & 0 & \tilde{\Xi}_{16} & 0 \\ * & \Xi_{22} & 0 & \theta_4 I & 0 & \tilde{\Xi}_{26} & \tilde{\Xi}_{27} \\ * & * & \Xi_{33} & 0 & 0 & P & 0 \\ * & * & * & \Xi_{44} & 0 & 0 & P \\ * & * & * & * & -Q & P & P \\ * & * & * & * & * & -P & 0 \\ * & * & * & * & * & * & -P \end{bmatrix}$$

$\tilde{\Xi}_{16} = A^T P - V^T B^T$,

$\tilde{\Xi}_{26} = V^T B^T$,

$\tilde{\Xi}_{27} = (A - LC)^T P$,

$Y = PL$.

By Lemma 3, we can obtain that (27) holds if

$$\breve{\Xi} = \begin{bmatrix} \tilde{\Xi}_1 + \Xi_2 & \hat{\Xi} \\ * & -\varphi\Phi - \varphi\Phi^T \end{bmatrix} < 0, \tag{28}$$

where

$$\hat{\Xi} = \varphi\Lambda_2 + \Lambda_3^T \Phi^T = \varphi \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ PB - B\Phi \\ 0 \end{bmatrix}}_{\Lambda_2} + \begin{bmatrix} -V^T \\ V^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Phi^{-T}\Phi^T = \begin{bmatrix} -V^T \\ V^T \\ 0 \\ 0 \\ 0 \\ \varphi(PB - B\Phi) \\ 0 \end{bmatrix}.$$

Now, by using Equation (3), (28) can be rewritten as

$$\breve{\Xi}=\begin{bmatrix}\tilde{\Xi}_1 & \hat{\Xi}\\ * & -\varphi\Phi-\varphi\Phi^T\end{bmatrix}+S_1\Delta_1(k)D_1 \\ +D_1^T\Delta_1^T(k)S_1^T+S_2\Delta_2(k)D_2+D_2^T\Delta_2^T(k)S_2^T<0, \tag{29}$$

where

$$D_1=\begin{bmatrix}N_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$
$$S_1^T=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & M_1^TP & M_2^TP & 0\end{bmatrix},$$

$$D_2=\begin{bmatrix}N_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$
$$S_2^T=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & -M_2^TY^T & 0\end{bmatrix}.$$

Now, using Lemma 2, (29) holds if

$$\begin{bmatrix}\tilde{\Xi}_1 & \hat{\Xi}\\ * & -\varphi\Phi-\varphi\Phi^T\end{bmatrix}+\eta_1D_1D_1^T \\ +\eta_1^{-1}S_1^TS_1+\eta_2D_2D_2^T+\eta_2^{-1}S_2^TS_2<0, \tag{30}$$

for any $\eta_1>0,\eta_2>0$.

By using Lemma 1, (15) imply (30) holds. So, we have that $\Pi<0$ . Then

$$\Delta V_k-\mu V_k-w_k^TQw_k<0, \tag{31}$$

which implies that

$$V_{k+1}<(1+\mu)V_k+w_k^TQw_k. \tag{32}$$

From (32), we get that

$$\begin{aligned}V_k&<(1+\mu)^kV_0+\sum_{i=0}^{k-1}(1+\mu)^{k-i-1}w_i^TQw_i\\ &\le(1+\mu)^NV_0+(1+\mu)^{k-1}\lambda_{\max}(Q)d^2\\ &\le(1+\mu)^N(V_0+\lambda_3d^2),\end{aligned} \tag{33}$$

From (17), we have

$$\begin{aligned}V_0&=x_0^TPx_0+e_0^TPe_0\\ &=\bar{x}_0^T\bar{R}^{\frac{1}{2}}\bar{R}^{-\frac{1}{2}}\bar{P}\bar{R}^{-\frac{1}{2}}\bar{R}^{\frac{1}{2}}\bar{x}_0\\ &\le\lambda_{\max}(\bar{R}^{-\frac{1}{2}}\bar{P}\bar{R}^{-\frac{1}{2}})\bar{x}_0^T\bar{R}\bar{x}_0\\ &\le\lambda_1c_1,\end{aligned} \tag{34}$$

$$\begin{aligned}V_k&=\bar{x}_k^T\bar{P}\bar{x}_k\\ &=\bar{x}_k^T\bar{R}^{\frac{1}{2}}\bar{R}^{-\frac{1}{2}}\bar{P}\bar{R}^{-\frac{1}{2}}\bar{R}^{\frac{1}{2}}\bar{x}_k\\ &\ge\lambda_{\min}(\bar{R}^{-\frac{1}{2}}\bar{P}\bar{R}^{-\frac{1}{2}})\bar{x}_k^T\bar{R}\bar{x}_k\\ &=\lambda_2\bar{x}_k^T\bar{R}\bar{x}_k,\end{aligned} \tag{35}$$

So, from (33)-(35), one get

$$\begin{aligned}\lambda_2\bar{x}_k^T\bar{R}\bar{x}_k&<(1+\mu)^N(V_0+\lambda_3d^2)\\ &<(1+\mu)^N(\lambda_1c_1+\lambda_3d^2),\end{aligned} \tag{36}$$

Using (14), we get that

$$\bar{x}_k^T\bar{R}\bar{x}_k<c_2.$$

According to Definition 1, the system (8) is FTB. This completes the proof.

**Remark 2** For a given $\mu$, the inequality (15) is linear matrices inequality which can be solved by MATLAB LMIs Toolbox to obtain matrices $P>0,Q>0,$ and matrices $Y,V,\Phi$.

### 3.2 $H_\infty$ Finite-time Boundedness

**Theorem 2** Given a scalar $\gamma>0$. Under Assumptions 1 and 2, the closed-loop system (8) is $H_\infty$ FTB with respect to $(c_1,c_2,\bar{R},N,\gamma)$, if there exist a known scalar $\varphi$, scalars $\mu\ge1$, $\varepsilon_1>0,\varepsilon_2>0,\varepsilon_3>0,\varepsilon_4>0$, $\eta_1,\eta_2$, symmetric matrices $P>0,Q>0$, and matrices $Y,V,\Phi$ such that the following inequalities hold:

$$(1+\mu)^N(\lambda_1c_1+\lambda_3d^2)<c_2\lambda_2, \tag{37}$$

$$\begin{bmatrix}\Theta_1 & \Theta_2\\ * & \Theta_3\end{bmatrix}<0, \tag{38}$$

where

$$\Theta_1=\begin{bmatrix}\tilde{\Theta}_{11} & 0 & \Theta_{13} & 0 & E^TF\\ * & \Theta_{22} & 0 & \Theta_{24} & 0\\ * & * & \Theta_{33} & 0 & 0\\ * & * & * & \Theta_{44} & 0\\ * & * & * & * & \tilde{\Theta}_{55}\end{bmatrix},$$

$$\Theta_2=\begin{bmatrix}\Theta_{16} & 0 & \Theta_{18} & 0 & 0\\ \Theta_{26} & \Theta_{27} & \Theta_{28} & 0 & 0\\ P & 0 & 0 & 0 & 0\\ 0 & P & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\Theta_3=\begin{bmatrix}\Theta_{66} & 0 & \Theta_{68} & \Theta_{69} & 0\\ * & \Theta_{77} & 0 & \Theta_{79} & \Theta_{7,10}\\ * & * & \Theta_{88} & 0 & 0\\ * & * & * & \Theta_{99} & 0\\ * & * & * & * & \Theta_{1010}\end{bmatrix},$$

$$\tilde{\Theta}_{11}=\Theta_{11}+E^TE,\quad \tilde{\Theta}_{55}=-\frac{\gamma^2}{(1+\mu)^N}I+F^TF,$$

and other parameters are given in (16). Furthermore, the controller gain is given by $K=\Phi^{-1}V$ and the observer gain is $L=P^{-1}Y$.

**Proof.** We show that for any $w_k\ne0$, under the zero-initial condition the output $z_k$ satisfies

$$\sum_{k=0}^{N}z_k^Tz_k<\gamma^2\sum_{k=0}^{N}w_k^Tw_k.$$

Choose the same Lyapunov function as (17) and define

$$J_k=\Delta V_k-\mu V_k+z_k^Tz_k-w_k^TQw_k. \tag{39}$$

Let

$$Q = \frac{\gamma^2}{(1+\mu)^N} I.$$

Then from (38), we have

$$J_k < 0. \tag{40}$$

From (40), it is obvious that

$$V_k < \sum_{i=0}^{k-1} (1+\mu)^{k-i-1} [w_i^T Q w_i - z_i^T z_i] + (1+\mu)^k V_0.$$

Under zero-initial conditions, $V_k \geq 0$ and $V_0 = 0$. Therefore, we have

$$0 < \sum_{i=0}^{k-1} (1+\mu)^{k-i-1} [w_i^T Q w_i - z_i^T z_i].$$

It follows that

$$\sum_{i=0}^{k-1} z_i^T z_i < \sum_{i=0}^{k-1} (1+\mu)^N \frac{\gamma^2}{(1+\mu)^N} w_i^T w_i.$$

It is deduced that

$$\sum_{i=0}^{N} z_i^T z_i < \gamma^2 \sum_{i=0}^{N} w_i^T w_i. \tag{41}$$

From (40), it is obvious that (31) holds. From Theorem 1, we get that the system (8) is FTB.

According to Definition 2, the system (8) is $H_\infty$ finite-time boundedness. The proof is completed.

**Remark 3** For a given $\mu$, the inequality (38) is a linear matrix inequality which can be solved by MATLAB LMIs Toolbox to obtain matrices $P > 0, Q > 0$, and matrices $Y, V, \Phi$.

**Remark 4** The system (8) considered in Theorem 2 has parameter uncertainty. However, systems that are often encountered in practical have no parameter uncertainty, so it is also necessary to study $H_\infty$ finite time boundedness for the system (8) with no parameter uncertainty. The following corollary gives a sufficient condition of $H_\infty$ finite time boundedness for (8) with $\Delta A(k) = 0, \Delta C(k) = 0$.

**Corollary 1** Given a scalar $\gamma > 0$. Under Assumptions 1 and 2, the closed-loop system (8) with $\Delta A(k) = 0$, $\Delta C(k) = 0$ is $H_\infty$ finite-time boundedness with respect to $(c_1, c_2, \bar{R}, N, \gamma)$, if there exist a known scalar $\varphi$, scalars $\mu \geq 1$, $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0, \varepsilon_4 > 0$, symmetric matrices $P > 0$, and matrices $Y, V, \Phi$, such that the following inequalities hold:

$$(1+\mu)^N (\lambda_1 c_1 + \lambda_3 d^2) < c_2 \lambda_2,$$

$$\begin{bmatrix} \hat{\Theta}_{11} & 0 & \Theta_{13} & 0 & E^T F & \Theta_{16} & 0 & -V^T \\ * & \Theta_{22} & 0 & \Theta_{24} & 0 & \Theta_{26} & \Theta_{27} & V^T \\ * & * & \Theta_{33} & 0 & 0 & P & 0 & 0 \\ * & * & * & \Theta_{44} & 0 & 0 & P & 0 \\ * & * & * & * & \tilde{\Theta}_{55} & 0 & 0 & 0 \\ * & * & * & * & * & -P & 0 & \Theta_{68} \\ * & * & * & * & * & * & -P & 0 \\ * & * & * & * & * & * & * & \Theta_{88} \end{bmatrix} < 0,$$

where $\hat{\Theta}_{11} = -(1+\mu)P + \theta_1 I + E^T E$, and other parameters are given in (16). Furthermore, the controller gain is given by $K = \Phi^{-1} V$ and observer gain is $L = P^{-1} Y$.

**Proof.** The proof is similar to the proof of Theorem 2, omitted here.

**Remark 5.** Some new criteria of observer-based finite-time $H_\infty$ control for discrete-time one-sided Lipschitz systems with uncertainties are given in Theorems 1 and 2. The main novelty of equation (26) is that the bilinear term $P^T BK$ can be eliminated by defining matrix variables $\Phi$ and $V$.

## 4. Numerical Example

In this section, a numerical example is given to show the application of the developed theory.

**Example 1.** Consider the nonlinear system (1) with the following parameters:

$$A = \begin{bmatrix} 0.1 & 0.01 \\ 0.02 & 0.1 \end{bmatrix}, B = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.12 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.3 & 0.1 \\ 0.3 & 0.1 \end{bmatrix}, \quad E = \begin{bmatrix} 0.02 & 0.1 \\ 0.2 & 0.1 \end{bmatrix},$$

$$F = \begin{bmatrix} 0.01 & 0.2 \\ 0.2 & 0.01 \end{bmatrix},$$

$$g(x_k) = \begin{bmatrix} 0.02\sin(x_k) \\ 0.02\sin(x_k) \end{bmatrix},$$

$$w_k = \begin{bmatrix} e^{-0.2k}\sin(k) \\ e^{-0.2k}\sin(k) \end{bmatrix}, \quad x_k = \begin{bmatrix} x_{1k} \\ x_{2k} \end{bmatrix}.$$

$$M_1 = \begin{bmatrix} 0.12 & 0.03 \\ 0.03 & 0.12 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0.11 & 0.02 \\ 0.02 & 0.11 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0.02 & 0.02 \\ 0.02 & 0.02 \end{bmatrix}.$$

Take $\varphi = 0.5, \eta_1 = \eta_2 = 2, c_1 = 0.2, \mu = 0.001, \gamma = 1, \rho = 0.01, \alpha = -0.3, \beta = -0.5, \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1.$

By using the MATLAB LMI toolbox, solving (37) and (38) leads to feasible solutions as follows:

$$P=\begin{bmatrix}0.2110 & 0.0051\\ 0.0051 & 0.1978\end{bmatrix}, V=\begin{bmatrix}-0.1363 & -0.0827\\ -0.0827 & -0.1214\end{bmatrix},$$

$$\Phi=\begin{bmatrix}8.9821 & -5.3997\\ -5.3997 & 7.8462\end{bmatrix}, Y=\begin{bmatrix}0.0633 & -0.3720\\ -0..3720 & -0.4188\end{bmatrix}.$$

The controller and observer gains are

$$K=\begin{bmatrix}-0.0367 & -0.0316\\ -0.0358 & -0.0372\end{bmatrix},$$

$$L=\begin{bmatrix}0.3458 & -1.7128\\ -1.8898 & -2.0734\end{bmatrix}.$$

According to Theorem 2, system (8) is $H_\infty$ finite-time boundedness with respect to $(0.2, 5.48, I_4, 40, 1)$. Figure 1 and Figure 2 show the state trajectories of $x_k$ and $\hat{x}_k$.

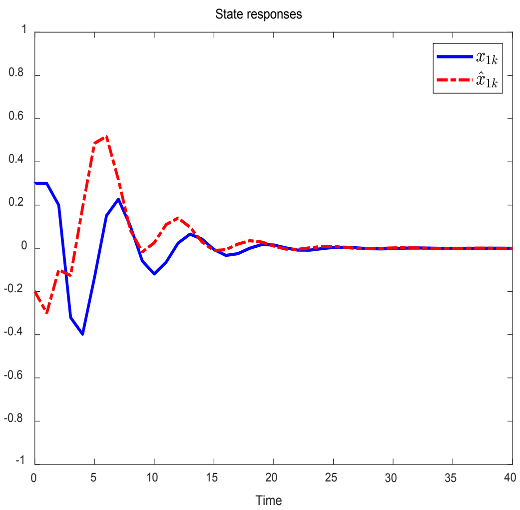


**Figure 1.** The trajectories of $x_{1k}$ and $\hat{x}_{1k}$.

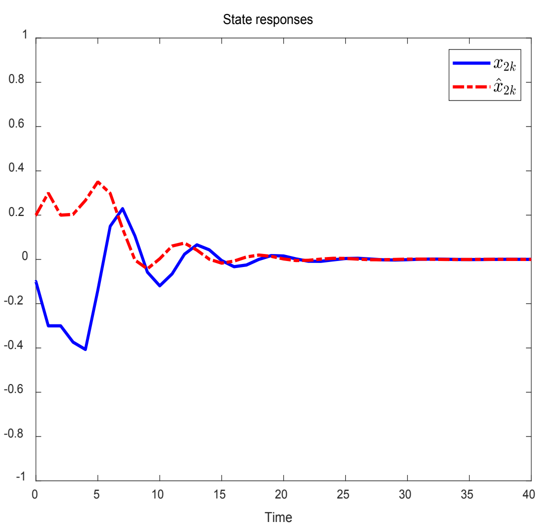


**Figure 2**. The trajectories of $x_{2k}$ and $\hat{x}_{2k}$.

The trajectory of $\bar{x}_k^T \bar{R} \bar{x}_k$ is shown in Figure 3. It can be seen from Figure 3 that when $k=1,2,\cdots,40$, the value of $\bar{x}_k^T \bar{R} \bar{x}_k$ is less than $c_2$.

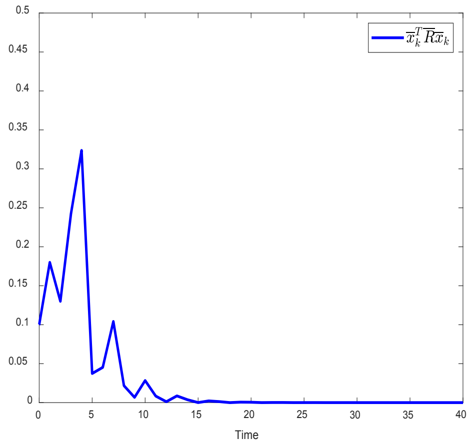


**Figure 3.** The trajectory of $\bar{x}_k^T \bar{R} \bar{x}_k$.

## 5. Conclusions

The paper discusses the observer-based finite-time $H_\infty$ control problem for a class of one-sided Lipschitz nonlinear discrete-time system with parameter uncertainties and external disturbances. By using one-sided Lipschitz condition and inner-bounded condition, a new criterion is obtained to ensure the closed-loop system is $H_\infty$ FTB. The observer and controller gains are designed. Finally, a numerical example is provided to demonstrate the applicability and reduced conservativeness of the presented results. Furthermore, this paper does not consider time-varying delay. Therefore, the problem of observer-based finite-time $H_\infty$ control for nonlinear discrete-time systems with time-varying delay can be investigated in the future work.

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## Conflicts of Interest

The author declares that they have no conflicts of interest to report regarding the present study.

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