REVIEW

The Voice of Physics in Finance: A Glance on the Theoretical Application of Heat Equation to Stock Price Diffusions

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ARTICLE INFO	ABSTRACT
	Stock price volatility is considered the main matter of concern within the investment grounds. However, the diffusivity of these prices should as well be considered. As such, proper modelling should be done for investors to stay healthy-informed. This paper suggest to model stock price diffusions using the heat equation from physics. We hypothetically state that, our model captures and model the diffusion bubbles of stock prices with a better precision of reality. We compared our model with the standard geometric Brownian motion model which is the wide commonly used stochastic differential equation in asset valuation. Interestingly, the models proved to agree as evidenced by a bijective relation between the volatility coefficients of the Brownian motion model and the diffusion coefficients of our heat diffusion model as well as the corresponding drift components. Consequently, a short proof for the martingale of our model is done which happen to hold.

1. Introduction

Quants and econophysicists have become more popular and recognizable since the 2008 economic meltdown. Their unwary contributions and dormant to finance and economics led to poor and bad behaviour of the used models. In his book [1], discusses about models behaving badly in the financial world. He posits that, bad models and confusing illusion with reality can lead to devastations and disasters in the Wall Street trading tables. In 2008, most firms closed while others experienced negative growth. Only Renaissance technologies managed to survive better in Wall Street as a Hedge Fund (thumbs up for Jim Simons-the mathematician). Therefore, such economic devastating effects can affect life too. Of course, models of financial nature do work well in most instances but its success is contingent upon no even economic operations. Such models are not good at capturing market complexities. This implies that, needs for super imposed models that capture and model such complexities with frequent and more accurate forecasts are really required, thus, the emergence, existence and relevancy of financial physics in finance. This not so long paper aims to provide some theoretically based applications of the Brownian motion heat equation to modelling stock price diffusions. We follow the theory of [2] on Brownian motion and stochastic processes guiding the stock market operations. Today, the concept of Brownian motion has been recognized in various ways as:

A process with independent homogenous increments whose paths are continuous. 2. The continuous time process which is the limit of symmetric random walks and

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lastly 3. the Markov process whose forward Kolmogorov equation is the heat equation. This study will focus on the third view, where we try to model the Markovness of stock prices using the diffusion heat equation. It is considerably valuable to analyze the paths taken by asset prices within markets especially to dynamic investors. This is because assets like stocks are dynamic and their prices are dynamic, continuous and random too. We theoretically study the diffusions associated with stock prices using the heat equation. Since its inception by [3], the heat equation has been exposed to wide and vast applications in the physics field not certainly in finance. In physics, we see the heat equation applied by [4] who investigated the effect of internal fins on flow pattern, temperature distribution and heat transfer between concentric horizontal cylinders for different fin orientations and fin tip geometry for Rayleigh numbers ranging from 103 to 106. They employed the two fin orientations used by [5]. Some other work is found in [6-8]. In addition, [9] studies the effect of magnetic field on the coupled heat and mass transfer by mixed convection in a linearly stratified stagnation flow in the presence of an internal heat generation or absorption. [10] studies thermal radiation effects on hydro magnetic free convection and flow through a highly porous medium bounded by a vertical plane surface, [11] considers the effect of radiation on unsteady natural convection in a two-dimensional participating medium between two horizontal concentric and vertically eccentric cylinders. We can only mention just a few. The bad part, which is the good part about our study, is the non-existence and or insignificant direct existence of both theoretical and empirical literature on the application of the heat equation to modelling asset price movements. Much of the work is concentrated on the application of the Black-Scholes model, see[12-14]. All the presented models in literature are without doubt applicable and more powerful. However, their immunity to financial diseases arising from rare events such as market crushes is really poor, hence the need of more power and more realistic forecasting models. Therefore, this study supports the use and application of physical models to modelling diffusions associated with asset prices, in particular of stocks. Heat diffusion equation is applied.

2. Main Results

We start by providing the widely used stochastic differential equation for stock price diffusions and the Brownian motion model before our main model. Stock prices are stochastic in nature and their uncertainty is not subject to vanishing in any way. As such, deterministic calculus and models cannot fully model the dynamics and diffusions associated with stock prices as they do not capture the randomness in association of stock prices. The widely used stochastic differential equation for time change of stock prices takes the following form as in [12].

$$d(S_t, t) = a(S_t, t)dt + b(S_t, t)dW_t \qquad (1)$$

Where, S_t is the stock is price, dW_t is innovation term representing unpredictable events that occur during the infinitesimal interval dt. Noting also that $a(S_t, t)$ and $b(S_t, t)$ are the drift and diffusion coefficients respectively. From the model (1) above, the innovation term, dW_t plays a vital role in explaining the randomness associated with stock prices, while their diffusions and dynamics are captured by $b(S_t, t)$. Moreover, [15] pointed out another related mean-reverting stochastic differential equation in the context of stock returns defined as:

$$dv_t = -\gamma (v_t - \theta)dt + \kappa\sqrt{v_t}dW_t \qquad (2)$$

The term θ is the long-time mean of ν, γ is the rate of relaxation to this mean, W_t is a standard Wiener process, and k is the variance noise parameter. This equation is well known in financial world as the CIR process and in mathematical statistics as the Feller process, (see, [16-17]). Now the standard geometric (multiplicative) Brownian motion model for stock prices in the Ito form is defined as follows:

$$dSt = \mu St\, dt + \sigma tSt\, dW \qquad (3)$$

The model is analogous to (1). The subscript t indicates time dependence, μ is the drift parameter, W_t is a standard random Wiener process, and σ_t is the time-dependent volatility. Thus, our heat equation diffusion model is constructed from the basis of equations (1), (2) and (3). We intend to move from being deterministic to being stochastic by introducing the drift and the volatility coefficients to our heat diffusion model. Our main modelling process is thus well explained below:

2.1 One Dimensional Heat Equation

We firstly present the physical heat equation in one dimension as below. The model is used to model one-dimensional temperature evolution [18].

$$u_t = a^2 u_{xx} \qquad (4)$$

The most important features of this equation are the second spatial derivative u_{xx} and the first derivative with respect to time, u_t, otherwise the derivation of the model is beyond the scope of this text. The reader is urged to consult [18-19]. The most important part of the model is the positive constant a^2 which is the diffusivity measure. Even though it is different from our final used model, its interpretation, relevancy and importance is similar.

It is however important to provide some contextual meaning of this simple one-dimensional equation. In

our modelling case, U is the price function of stock assets which is subject to market forces from which the diffusion/volatility component, a^2 gets support. Note that, the model is time inhomogeneous in the sense that, it depends on time t. The equation can as well be extended to a two dimensional function, see [20]. Next we provide our main model which better models the diffusions of the stock prices in a continuous time space. Note that the model originates from the one-dimensional heat equation defined above.

2.2 Main Model

This section presents the main diffusion model used in this study. The model takes the nature of the heat equation and is presented below:

$$\partial_t U = -\sum_{j=1}^{n} \left(\partial_{x_j} a_j(x) u\right) + \sum_{j,k=1}^{n} \left(\partial_{x_j} \partial_{x_k} (u_{j,k}(x) u)\right) \qquad (5)$$

The matrix of the diffusion coefficients, μ is related to the volatility function in the standard stochastic differential equation (1) b by the following:

$$\mu(x) = b(x) \times b^*(x) \qquad (6)$$

Here b^* is the transpose of the matrix b and the coefficients $a_j(x)$ correspond directly to a in the standard equation in (1) above. We are not as well interested in the derivation of the model in (5). The reader should note that the function U is the probability density function which in this case is the price probability density function. As such, if we integrate it over x, we should obtain something which is independent of t. By such, we can safely consider it as a sufficient price function. Note that, we consider our stock prices as martingales and as Markovian. Our approach as stated earlier is relying on the propositions of Louis Bachelier which one of them states that stock prices exhibit some patterns of Markov nature. In actual fact, the partial differential equation in (5) is a pure martingale. This is following the derived conditional expectation or the drift that $a_j = 0$, and if $a_j = 0$ then prices do not change over time. Non-martingale property is of course important in analysing stock prices. We will not get into detail of these. Our mission is on the diffusivity modelling of stock prices which we are now turning into in the below short paragraphed section. But before that, we provide below some evidence of the existence of the non-martingale property in equation (5). We take the expectation as follows:

If $a_j = 0$ then we have $\partial_t E(X_t) = \partial_t \int x u(x,t) dx$ which leads to the following:

$$\int \frac{x}{2} \sum_{j,k=1}^{n} \left(\partial_{x_j} \partial_{x_k}(u_{j,k}(x) u)\right) d = 0 \qquad (7)$$

2.3 Diffusivity of Stock Prices

The main underlying aim of this paper is to model stock price diffusions within stock markets over a continuous time space. From the standard stochastic differential equation (SDE) in (1) above, stock volatility matters much to investors especially the risk averse ones. This implies that, whatever the volatility coefficient say or suggest, investors do pay attention and respect those exhibitions by responding accordingly. The main reason is that, they want to secure their investment payoffs. It is the stock price volatility and diffusion behaviour of some sort that explains the investment status in stock assets. However, little consideration seems to be paid on the links and relations between volatility and stock diffusions. This paper is prepared for that task. We shall in this section compare the standard equation in (1) with our diffusion equation in (5) above. We aim to compare the volatility coefficients and the diffusion coefficients of the models. Such, relation is considered to be helpful in explaining and establishing the relevancy of our main model in stock markets. The essence goes as:

Using the relation in (6), we first make a supposition that m=n=1 such that, μ , depend on b in the relation given by $\mu=b^2$. Generally speaking, we need a matrix which is analogous to the relation, $\mu=b^2$, such that we derive a square n×n matrix from n×m of b. This is well explained in a much simpler manner in the relation (6). Further, using the moment-matching approach we can safely and clearly compare our models and their corresponding volatility and diffusion components. Suppose b is a constant and, a=0, then x(0)=0 implies that z(0)=0 and we get X(t)=bZ(t). Thus, it follows that, $Cov(X)=E[X(t)X^*(t)]=bb^*t$. Interestingly from (5) we get the expectation

$$\partial_t E(XX^*) = \partial_t \int xx^* U(x,t) dx = \mu \qquad (8)$$

which agrees with the relation in (6) We further provide some insights on the drift component. The drift component in (1) is given by aXdt and it corresponds to $\partial_x au$ in (5). This can easily be followed if a is a constant and if b=0. The other interesting corollary is the negative transition in the probability density function of stock prices by a factor u(x,t)=u(x-at). We subsequently provide some important results below which are informative.

3. Observational Results

We provide some simple observational results following our analysis and modelling of stock prices. We note that, stock prices undergo some normal jumps which are better explained by a diffusion process defined above. The fundamental theorem of Levy we note that there

is no difference between the Wiener process and the diffusion process in their applications. The two can be used interchangeably. Thus, we note that, stock prices follow the Wiener process, when, $Z(0)=0$. Additionally, we note that the information from the observed diffusion bubbles associated with stock prices are helpful on planning in terms of asset allocation (portfolio creation), asset pricing and valuation, risk management and proper investment choice making. We noted some modelling related problems for heat equations which must be factored into account whenever our modelling makes use of the heat family of equations. Some of them include the infinite speed propagation with limited spreading, time step constraints for explicit difference methods and the smoothing property. Such features should be considered when dealing with our model defined in (5) for non-misleading results. We thus put forward that marching methods and finite differencing can be well used to solve heat diffusion equations.

4. Conclusions

We conclude that stock price diffusions can well be explained by the heat diffusion equation. Our implemented model is considered worthwhile because it is in support with the standard geometric Brownian motion model. By comparing the drift and the volatility components of (1) to the drift and diffusion properties of (5) and the relation (6) we note that these are in agreement and thus our model can better explain the diffusivity of stock prices. Our model too can be used to explain the jumps associated with the stock prices as it provides a good base for analysing stock price paths in the market under a continuous time space. Such normal and systematic jumps are without doubt informative to investors and thus should be considered. Therefore, we note that our model is more applicable and informative. This surfaces the value and power of physical models in the financial world. We thus make a recommendation of their use in financial modelling. The reader is finally recommended to read the book by [1], "Models Behaving Badly" for more on such modelling phenomena.

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