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REVIEW

Review of Proposed Stress-dilatancy Relationships and Plastic Potential Functions for Uncemented and Cemented Sands

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ABSTRACT

Stress-dilatancy relationship or plastic potential function are crucial components of every elastoplastic constitutive model developed for sand or cemented sand. This is because the associated flow rule usually does not produce acceptable outcomes for sand or cemented sand. Many formulas have been introduced based on the experimental observations in conventional and advanced plasticity models in order to capture ratio of plastic volumetric strain increment to plastic deviatoric strain increment (i.e. dilatancy rate). Lack of an article that gathers these formulas is clear in the literature. Thus, this paper is an attempt to summarize plastic potentials and specially stress-dilatancy relations so far proposed for constitutive modelling of cohesionless and cemented sands. Stress-dilatancy relation is usually not the same under compression and extension conditions. Furthermore, it may also be different under loading and unloading conditions. Therefore, the focus in this paper mainly places on the proposed stress-dilatancy relations for compressive monotonic loading. Moreover because plastic potential function can be calculated by integration of stress-dilatancy relationship, more weight is allocated to stress-dilatancy relationship in this research.

1. Introduction

Variations of soil and rock volume may be caused by changes in the stress and deformation (mechanical effects) and changes in the water content, the temperature and other reasons (non-mechanical effects) [54]. The tendency of soil or rock to change volume while shearing is called dilatancy [18]. Dilatancy of sand and cemented sand is a crucial phenomenon which differentiates their plastic response from that of metal. That is, dilatancy is one of features of sand and cemented sand that can not be simulated appropriately by the traditional plasticity developed initially for metals. Usefulness of any elastoplastic constitutive model of sand or cemented sand may highly depends on successful implementation of dilatancy response. It is clear that loose sands or weakly cemented sands tend to develop positive pore pressure under undrained shear or to decrease volume under drained shear conditions. Conversely, dense sands and cemented sands tend to develop negative pore pressure in the course of shear or to increase in volume under drained shear conditions [7]. That is, dense sands dilate as deviator stress increases until failure while loose sands decreases in volume with progress of shear loading until failure. This is why dilatancy theory is normally discussed in the context of dense sands. However there is

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usually nothing in proposed stress-dilatancy relations that limit their application to dense sands [4].

Note that in loose sands or weakly cemented sands, zero dilatancy takes places at large strains, but in dense sands, it happens twice during shearing: once temporary at phase transformation when response changes from contractive to dilative, and once at large strains when shearing continues at constant volume [16, 30].

The geotechnical literature uses the terms dilation and dilatancy interchangeably as we did in the last paragraphs. They are usually stated as an angle compared to friction angle. However, we can divide these two dilatancy definitions into two categories: 1) absolute definition: dilation is the volumetric strain change occurred since the initial condition 2) rate definition: dilation is the ratio of volume change increment relative to shear strain increment [18]. We use the rate definition of dilation in next sections.

Note that the stress-dilatancy equations and plastic potential functions are used extensively in this article. The plastic potential defines the direction of plastic strain increments. The plastic potential is usually expressed using a stress-dilatancy equation which relates the dilatancy rate to the stress ratio [21].

2. Background

According to the classical plasticity, there exist a plastic potential function such that normal to the function shows components of plastic strain increment. This is to say [12, 67, 68]:

$$\dot{\varepsilon}_{ij}^{p} = \lambda \frac{\partial g}{\partial \sigma_{ij}} \tag{1}$$

where $\dot{\varepsilon}_{ij}^{p}$ is plastic strain increment tensor, g is plastic potential function, σ_{ij} is stress tensor and λ is plasticity multiplier which is calculated using consistency condition.

This rule is called flow rule. Flow rule implies that the derivative of the plastic potential function is important in constitutive modelling rather than the function itself.

Stress-dilatancy relationship in triaxial space also is usually expressed as follows [4,9,35]:

$$d = \frac{\dot{\varepsilon}_{P}^{p}}{\dot{\varepsilon}_{q}^{p}} = \frac{\dfrac{\partial g}{\partial p'}}{\dfrac{\partial g}{\partial q}} \tag{2}$$

where d is the dilatancy rate, $\dot{\varepsilon}_{p}^{p}$ and $\dot{\varepsilon}_{q}^{p}$ are volumetric and deviatoric plastic strain increments accordingly, p' and q are mean effective stress and deviator stress, respectively.

Analysis of direct shear box test of sand by Taylor [57] implies that input work is entirely dissipated in friction. Therefore, we can equal plastic input work to the dissipated energy in friction for triaxial conditions as follows [6,7,34]:

$$\begin{aligned} p'\dot{\varepsilon}_{P}^{p} + q\dot{\varepsilon}_{q}^{p} &= Mp'\dot{\varepsilon}_{q}^{p} && \text{for triaxial compression} \\ p'\dot{\varepsilon}_{P}^{p} + q\dot{\varepsilon}_{q}^{p} &= -Mp'\dot{\varepsilon}_{q}^{p} && \text{for triaxial extension} \end{aligned} \tag{3}$$

M denotes the slope of critical state line in $p' - q$ space.

Arrangement of equation (3) gives rise to a very fundamental form of dilatancy equation as follows [6,19,34]:

$$d = \frac{\dot{\varepsilon}_{P}^{p}}{\dot{\varepsilon}_{q}^{p}} = \begin{cases} M - \eta & for\ compression \\ -M - \eta & for\ extension \end{cases} \tag{4}$$

η is stress ratio which is defined as q/p'.

Different values of M are used for triaxial compression and triaxial extension as follows:

$$M = \begin{cases} \dfrac{6 \sin\phi_{cv}}{3 - sin\phi_{cv}} & for\ compression \\ \dfrac{6 \sin\phi_{cv}}{3 + sin\phi_{cv}} & for\ extension \end{cases} \tag{5}$$

ϕ_{cv} is the constant volume (critical state) fiction angle.

From the above equation, it can be realized that M (the critical state stress ratio) for compression is greater than M for extension if the critical state friction angle is the same for both compression and extension.

In order to arrive in better match for experimental observations, a constant A is often introduced in equation (4) in the following form [15,16,69]:

$$d = \frac{\dot{\varepsilon}_{P}^{p}}{\dot{\varepsilon}_{q}^{p}} = A(M - \eta) \tag{6}$$

A is a material parameter which may be different for triaxial compression and extension loading and η is current stress ratio. A should be determined by curve fitting of equation (6) against total volume change [15,16].

Equation (4) can also be written in the form of equation (7) as follows [7]:

$$d = \frac{\dot{\varepsilon}_{P}^{p}}{\dot{\varepsilon}_{q}^{p}} = (M - \beta\eta) \tag{7}$$

β is a constant.

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Rowe [47] derived the stress-dilatancy relation for soils using minimum energy consideration for particle sliding as follows [4,41]:

$$\frac{\sigma_1'}{\sigma_3'} = K\left(1-\frac{\dot{\varepsilon}_V}{\dot{\varepsilon}_1}\right) \quad for\ compression \tag{8}$$

$$\frac{\sigma_1'}{\sigma_3'} = \frac{K}{\left(1-\frac{\dot{\varepsilon}_V}{\dot{\varepsilon}_1}\right)} \quad for\ extension \tag{9}$$

where K is a parameter which depends on the failure friction angle.

With presuming a constant failure friction angle and neglecting elastic strains, Wood [66] formulated Rowe's stress-dilatancy relation as follows [15, 62, 41]:

$$\begin{cases} d = \dfrac{\dot{\varepsilon}_p^p}{\dot{\varepsilon}_q^p} = \dfrac{9(M_c-\eta)}{9+3M_c-2M_c\eta} & for\ compression \\ d = \dfrac{\dot{\varepsilon}_p^p}{\dot{\varepsilon}_q^p} = \dfrac{9(M_e-\eta)}{9-3M_e-2M_e\eta} & for\ extension \end{cases} \tag{10}$$

where M_c and M_e are critical state stress ratio in triaxial compression and extension, respectively.

In addition to the constant critical state stress ratio, the variable phase transformation stress ratio (i.e. M_{PT}) [for example see 16, 42,43] and the failure stress ratio (i.e. M_f) have been used in equations (4), (6) and (10) as well. M_{PT} is a value corresponding to a temporary zero dilatancy which is calculated similar to equation (5) with replacement of the critical state friction angle by the phase transformation friction angle [6,69]. M_f also is not constant and evolves towards the critical state stress ratio with plastic shearing [4,16]. Replacement of the critical state stress ratio with M_{PT} or M_f is performed in order to account for the effect of pressure and density dependency of the stress-dilatancy relationship of sands [15].

3. Stress-dilatancy Relationships and Plastic Potential Functions

3.1 Cohesionless Sands

Proposed stress-dilatancy relationships and plastic potential functions are presented in a historical sequence. Most of formulas which are presented here are stress-dilatancy relationships. However, some proposed constitutive models have used a plastic potential function to define the direction of plastic flow as well. A summary of these stress-dilatancy relationships and plastic potential functions of uncemented sands with striking highlights is given in table 3 (see appendix section).

(1) Nova and Wood [35]:

Nova and Wood [35] used the following equation as their stress-dilatancy relationship:

$$d = \frac{M-\eta}{\mu} \tag{11}$$

where M and μ are positive constants and d is the dilatancy rate. At critical state, $\eta=M$ and hence dilatancy rate becomes zero which implies no volume change at the critical state. ($-1/\mu$) is the slope of the linear relation between $\dot{\varepsilon}_p^p/\dot{\varepsilon}_q^p$ and η.

Cui and Delage [8] applied the same equation as above in their constitutive model.

Plastic potential function associated with this stress-dilatancy relationship is as follows:

$$g(p',q) = q - \frac{Mp'}{1-\mu}\left[1-\mu\left(\frac{p'}{p_{ug}'}\right)^{(1-\mu)/\mu}\right] \tag{12}$$

where p_{ug}' is value of the isotropic pressure when $\eta=M$. Note that when $\mu=1$, the equation reduces to that of the Cam Clay model.

(2) Nova and Wood [36]:

One of the oldest and most popular stress-dilatancy relationships proposed for cohesionless sands was first suggested by Nova and Wood [36]. It was formulated as follows:

$$d = (1+\alpha_g)(M-\eta) \tag{13}$$

where $\eta=q/p'$ is stress ratio, M is the critical state stress ratio and a_g is a material parameter.

Similar stress-dilatancy relationships were used by Pastor et al. [37], Haeri and Hamidi [13], Hamidi and Yarbakhti [14] and Kong et al. [25].

The above relationship results in the following plastic potential function:

$$g(p',q) = q - Mp'\left(1+\frac{1}{\alpha_g}\right)\left[1-\left(\frac{p'}{p_g'}\right)^{\alpha_g}\right] \tag{14}$$

where p_g' is a size parameter which determines size of the plastic potential function.

(3) Jefferies [17]:

Jefferies [17] used the below relationship in his critical state constitutive model to represent the stress-dilatancy behavior. It appears that the formula was proposed based on Nova and Wood [35]'s relation as follows:

$$d = \frac{M - \eta}{1 - N} \tag{15}$$

where N is a material parameter and d is the dilatancy rate which is defined as plastic volumetric strain increment to plastic deviatoric strain increment.

If N=0 then the above equation reduces to the familiar Cam Clay model. Note that Andrade [2] used the same flow rule in his constitutive model.

Jefferies obtained the plastic potential function by integration of his stress-dilatancy relationship and then considered the yield function equals to the plastic potential function (associated flow rule).

Jefferies stated that their stress-dilatancy relationship and constitutive model in general capture the influence of void ratio and mean effective stress on sand response but neglect the influence of fabric (i.e. grain contact arrangement or geometrical packing of particles). He also claimed that the bedding direction during sample preparation pronouncedly affects stress-dilatancy relationship.

Jefferies suggested in his model that an infinite number of isotropic consolidation lines exist for sands which prevents the direct coupling of the yield surface size to the void ratio. This was a novelty in his model which resulted in change of focus from the normally consolidation line to the critical state line in definition of the hardening law. That is, he applied a shear hardening law instead of a volumetric hardening law.

(4) Lagioia et al. [27]:

Lagioia et al. [27] proposed their stress-dilatancy relationship such that the proposed formula fulfills the two following requisites:

$$\begin{cases} \eta \to 0 & \Rightarrow d \to \infty \\ \eta = M & \Rightarrow d = 0 \end{cases} \tag{16}$$

This implies that when a material is sheared isotropically (η=0), only plastic volumetric strain occurs and also when the stress ratio reaches the critical state value of M in compressive loading, the critical state condition must meet [27]. The following stress-dilatanccy relation satisfies the two aforementioned conditions:

$$d = \mu_g (M - \eta)(\frac{\alpha M}{\eta} + 1) \tag{17}$$

α and μ_g are model constants. η is the current stress ratio and M is critical state stress ratio which is deifned based on equation (5).

Similar to Jefferies [17], Lagioia et al. [27] integrated the above equation and considered it equals to the yield function (associated flow rule).

Note that similar stress-dilatancy equation was later used by DeSimone and Tamagnini [9].

(5) Yu [71]:

Yu [71] used the well-known Rowe's stress-dilatancy relationship (equation (10)) in his proposed constitutive model named CASM to relate stress ratio and dilatancy rate. Yu [71] stated that despite immense effort to develop a better stress-dilatancy relation for sands compared to Rowe's original or modified relationship, little progress appears to have been made in this area. Based on Rowe's stress-dilatancy relation, Yu [71] and Yu et al. [72] obtained the following relation for the plastic potential of sands and clays:

$$g(p^{'},q) = 3M(lnp^{'} - ln\beta_g) + (3 + 2M)\ln\left(\frac{3p^{'} + 2q}{p^{'}}\right) - (3 - M)\ln\left(\frac{3p^{'} - q}{p^{'}}\right) \tag{18}$$

The size parameter β_g is a dummy parameter as only the derivative of the plastic potential function with respect to $p^{'}$ and q is important.

Note that the adopted stress-dilatancy relation in CASM is non-associated flow rule as the plastic potential function is not the same as the proposed yield function in this model.

(6) Schanz et al. [15]:

Schanz et al. [15] used the following equations as the flow rule:

$$sin\psi_m = \frac{sin\varphi_m - sin\varphi_{cv}}{1 - sin\varphi_m sin\varphi_{cv}} \tag{19}$$

where Ψ_m is the mobilized dilatancy angle, φ_m is the mobilized friction angle and φ_{cv} is the critical state friction angle which is a material constant. φ_m is calculated by below equation:

$$sin\varphi_m = \frac{\sigma^{'}_1 - \sigma^{'}_3}{\sigma^{'}_1 + \sigma^{'}_3 - 2coh.cot\varphi_p} \tag{20}$$

where φ_p is the failure friction angle. This equation can easily be proved by shape of a Mohr-Coulomb failure envelope.

The above equations correspond to the famous Rowe's stress-dilatancy theory [15].

Note that the critical state dilation angle can be calculated by the failure friction and dilation angles as follows:

$$sin\psi_{cv} = \frac{sin\varphi_p - sin\psi_p}{1 - sin\varphi_p sin\psi_p} \tag{21}$$

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The above definition of the flow rule is equivalent to the definition of the plastic potential function in the form of below equation [15]:

$$g = \frac{\left(\sigma_1^{'} - \sigma_3^{'}\right)}{2} - \frac{\left(\sigma_1^{'} + \sigma_3^{'}\right)}{2} sin\psi_m \qquad (22)$$

(7) Wan and Guo [61]:

Wan and Guo [61] presented a fabric dependent stress-dilatancy equation as follows:

$$d = \frac{-4}{3} \frac{\left(sin\varphi_m - sin\varphi_f\right)}{\left(1 - sin\varphi_f sin\varphi_m\right)} \qquad (23)$$

Depending on magnitude of the characteristic friction angle (φ_f) relative to that of the mobilized friction angle (φ_m), the dilatancy rate changes its sign. This enables the model to capture the change in volumetric response from contractive to dilative which usually observed in dense sands experimentally.

The characteristic and mobilized friction angle in this model are defined as follows:

$$sin\varphi_f = \frac{X\left(\frac{F_{33}}{F_{11}}\right) + \gamma^{P^*}}{a + \gamma^{P^*}} \left(\frac{e}{e_{cr}}\right)^{\alpha} sin\varphi_{cv} \qquad (24)$$

where F_{11} and F_{33} are fabric tensor components in the axial and radial directions, respectively.γ^{P^*} is the true shear strain. X, a and α are material constants. e is the current void ratio and e_{cr} is the critical state void ratio.

$$sin\varphi_m = \frac{\gamma^{P^*}}{a + \gamma^{P^*}} \left(\frac{e}{e_{cr}}\right)^{-\beta_1} sin\varphi_{cv} \qquad (25)$$

where β_1 is a material parameter.

(8) McDowell [31]:

McDowell [31] proposed the following stress-dilatancy relation for sand in his constitutive model:

$$d = \frac{M^{b+1} - \eta^{b+1}}{\eta^b} \qquad (26)$$

where d is dilatancy ratio, $\eta = q/p^{'}$ is current stress ratio and M is the critical state stress ratio and b is a material parameter.

The plastic potential associated with this stress-dilatancy equation is defined as follows:

$$g\left(p^{'}, q\right) = \eta - M\left[(b+1)\ln\left(\frac{p_p^{'}}{p^{'}}\right)\right]^{\frac{1}{b+1}} \qquad (27)$$

$p_p^{'}$ is the value of $p^{'}$ at q=0

McDowell's stress dilatancy equation fulfills the necessary conditions of zero shear strain under isotropic conditions $\left(\eta = 0 \rightarrow \varepsilon_q^p = 0 \rightarrow d = \infty\right)$ and infinite shear strain with zero volumetric strain at the critical state . $\left(\eta = M \rightarrow \varepsilon_q^p = \infty \;\&\; \varepsilon_p^p = 0 \rightarrow d = 0\right)$ This approach is identical to that suggested by Lagioia et al. [27].

(9) Russell and Khalili [48]:

Russell and Kalili [48] used the following equation as the streee-dilatancy relation:

$$d = M_f - \eta = \left(1 + k_d \xi\right) - \eta \qquad (28)$$

where k_d is a material paramter and ξ is the state parameter.

M_f is used in above equation instead of M to capture the dependency of d on density and confining pressure through k_d and ξ.

Russell and Kalili [48]'s model was developed in a critical state framework. Certain features of the model are linked to the state parameter (ξ), which is a dimensionless parameter defined as the vertical distance between the current state and the critical state line in the $e - p^{'}$ plane. The state parameter is positive on the loose (wet) side of the critical state line (CSL) and negative on the dense (dry) side of CSL, and expressed as:

$$\xi = e - e_{cs} \qquad (29)$$

where e is the void ratio at the current stress state ($p^{'}$), and e_{cs} the void ratio at the critical state corresponding to $p^{'}$.

(10) Been and Jefferies [4]:

Been and Jefferies [4] summarized the existing stress-dilatancy relations according to the table 1 and table 2:

Table 1. Summary of stress-dilatancy relationships [4]

Theory	Relationship
Cam Clay	$d=M_f-\eta$
Modified Cam Clay	$d=(M^2_f-\eta^2)/2\eta$
Jefferies ([17]	$d=(M_f-\eta)/(1-N)$
Li and Dafalias [29]	$d=d_0(M_f-\eta)/M$
Rowe [47]	$d=9(M_f-\eta)/(9+3M_f-2M_f\eta)$

Table 2. Summary of proposed relationships for M_f[4]

Authors	Relationship	Comments
Manzari and Dafalias [33]	$M_f=M+m\xi$	ξ is the state parameter.

Li and Dafalias [29]	$M_f=M\exp(m\xi)$	ξ is the state parameter.
Jefferies and Schuttle [19]	$M_f=M-\lvert\xi_i\rvert$	$\xi_i=\xi-\lambda(1-\eta/M)$
Wang et al. [64]	$M_f=M_0+(M-M_0)I_p$	$\xi=\lambda\ln(I_p)$

Note: λ is slope of the critical state line. m and M_0 are positive model constants.

(11) Dafalias and Manzari [33]:

Dafalias and Manzari [33] proposed their stress-dilatancy equation based on difference between the current stress ratio η and the dilatancy stress ratio M_{PT} which is known as the phase transformation stress ratio. The suggested relationship is as follow:

$$d = A_d(M_{PT} - \eta) \tag{30}$$

where A_d is material parameter.

For dense sands under small shear strains, $\eta < M_{PT}$ which leads to $d > 0$ (compressive behavior). However, with progress of loading, η becomes equal to M_{PT} at a point which results in d=0 (temporary steady state response). Further shearing leads to $\eta > M_{PT}$ which results in $d < 0$ (dilative behavior). Ultimately, the phase transformation stress ratio becomes equal to the current stress ratio at the critical state which leads to zero dilatancy rate (permanent steady state response). Note that the phase transformation state is never reached during shearing of loose sands and η is always smaller than M_{PT} leading to prediction of permanent contraction during shearing.

Dafalias and Manzari made A_d fabric dependent. To avoid lengthy discussion, fabric dependency of the stress-dilatancy relationship is not discussed here. Interested reader is referred to Dafalias and Manzari [33].

(12) Khalili et al. [23]:

The stress-dilatancy equation which was proposed by Khalili et al. [23] is similar to that of Russell and Khalili [48]. The proposed relation is as follows:

$$d = M_f - \eta = (1 + k_d\xi)M - \eta \tag{31}$$

where k_d is a material constant, ξ is the state parameter and M is the critical state stress ratio.

The variable M_f is used in place of the constant M to account for the dependency of d on void ratio (or density).

Integration of the above stress-dilatancy relation with respect to p' and q results in the following plastic potential function:

$$g(p', q, p_0) = q + (1 + k_d\xi)Mp'\ln\frac{p'}{p_0} \tag{32}$$

As mentioned already ξ is a dimensionless parameter defined as the vertical distance between the current state and the critical state line in the v $vs.$lnp' plane. It is expressed as:

$$\xi = \nu - \nu_{cs} \tag{33}$$

where v=1+e is the specific volume at the current stress state (p'), e is the void ratio, and v_{cs} is the specific volume at the critical state corresponding to p'.

(13) Imam et al. [16]:

Imam et al. presented their stress-dilatancy relation based on concept of the phase transformation stress ratio M_{PT} as follows:

$$d = A(M_{PT} - \eta) \tag{34}$$

A is calculated for triaxial compression and triaxial extension, respectively, as follows:

$$\begin{cases} A_c = \dfrac{9}{9 + 3M_{PT,c} - 2M_{PT,c}\eta} \\ A_e = \dfrac{9}{9 - 3M_{PT,c} - 2M_{PT,c}\eta} \end{cases} \tag{35}$$

where

$$\begin{cases} M_{PT,c} = \dfrac{6\sin\varphi_{PT,c}}{3 - \sin\varphi_{PT,c}} \\ M_{PT,e} = \dfrac{6\sin\varphi_{PT,e}}{3 + \sin\varphi_{PT,e}} \end{cases} \tag{36}$$

and

$$\begin{cases} \sin\varphi_{PT,c} = \sin\varphi_{cs} + k_{PT}\xi \\ \sin\varphi_{PT,e} = \sin\varphi_{cs} + k_{PT}\xi + a_{PT} \end{cases} \tag{37}$$

where k_{PT} and a_{PT} are material parameters, φ_{cs} is the critical state friction angle, φ_{PT} is the phase transformation friction angle, M_{PT} is the phase transformation stress ratio and ξ is the state parameter.

The plastic potential function associated with the above flow rule is obtained by the following relationship:

$$g(p',q) = \eta - M_{PT}\left(\frac{A}{A-1}\right)\left[1 - \left(\frac{p'}{p'_0}\right)^{A-1}\right] \qquad for\ A \neq 1 \tag{38}$$

in which p'_0 is a measure of the size of the plastic potential function.

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The same stress-dilatancy equation was used by Tasiopoulou and Gerolymos [56] in their model. They, however, assigned a value of one for *A*.

(14) Ling and Yang [30]:

Ling and Yang [30] used a combination of work of Nova and Wood [36] and Li and Dafalias [29] to define their stress-dilatancy relation as follows:

$$d = \left(1 + \alpha_g\right)(M_f - \eta) \tag{39}$$

$$M_f = M \exp{(m\xi)} \tag{40}$$

where α_g and m are positive constants and ξ is the state parameter. α_g and m can be determined using the volumetric strain vs. deviator strain curve.

The difference between the above equation with that of Nova and Wood is that a variable stress ratio M_f (which changes with the void ratio and mean effective stress) has been used rather than the constant critical state stress ratio *M*.

The state parameter at the phase transformation for dense sands temporary becomes zero which enables the constitutive model to capture the phase transformation state. However for loose sands, the void ratio is always greater than the critical state void ratio which leads to a positive state parameter. The net result is that M_f is always greater than the current stress ratio which implies that the dilatancy rate is always positive for loose sands (i.e. permanent contractive behavior) except at the critical state in which the dilatancy rate becomes zero. This implies that no volume change occurs at the critical state neither for loose sands nor dense sands because at the critical state, the state parameter is zero which leads to $M_f=M=\eta$.

Liu et al. [32] considered the similar stress-dilatancy equation in their model. However, they replaced M_f with the phase transformation stress ratio.

(15) Sasiharan [49]:

Sasiharan [49] suggested the following stress-dilatancy equation in his anisotropic sand model:

$$d = \frac{M^2 + \alpha_g^2 - \eta^2}{2(\eta - \alpha_g)} \tag{41}$$

where is fabric anisotropy parameter which is defined according to the following formula:

$$\alpha_g = \left(\frac{e}{1+e}\right)\beta \tag{42}$$

in which e is current void ratio and β is a material parameter. Thus, the fabric anisotropy parameter evolves with the void ratio.

The plastic potential function corresponds to the above stress-dilatancy relationship is as follows:

$$g(p',q) = \frac{p'}{p'_{cg}} - \frac{M^2}{M^2 + (\eta - \alpha_g)^2} \tag{43}$$

where p'_{cg} is the value of p' when $\eta=\alpha_g$.

Similar to Jefferies's approach and conventional Cam Clay model, Sasiharan considered the resulting plastic potential function, obtained by integration of the flow rule, identical to the yield function.

If the fabric anisotropy parameter α_g is zero, the yield and plastic potential surfaces will be an ellipse that is centered along the p' axis as in modified Cam Clay model.

Sasiharan [49] stated that the above stress-dilatancy relation is not appropriate for anisotropic sands and suggested the following flow rule instead:

$$d = \frac{M^2 - (\eta - \alpha_g)^2}{2(\eta - \alpha_g)} \tag{44}$$

$\eta = (M^2 + \alpha_g^2)^{\frac{1}{2}}$ for the first flow rule and $\eta = \pm M + \alpha_g$ for the second flow rule constitute the phase transformation line (i.e. zero dilatancy line) on which zero plastic strain increment occurs.

Note that similar stress-dilatancy relationships were mentioned in Muhunthan and Sasiharan [34].

(16) Tasiopoulou and Gerolymos [55]:

Tasiopoulou and Gerolymos [55] presented their stress-dilatancy equation as follows:

$$d = M_{PT} - \zeta M_s \tag{45}$$

where M_{PT} is the phase transformation stress ratio, M_S is ultimate strenght stress ratio in q-p' space, and ζ is the hardening parameter which evolves with shearing and is defined as:

$$\zeta = \left|\frac{q}{M_s p'}\right|^n \tag{46}$$

where n is an exponential parameter which controls the distance of the current stress state from the failure line.

The ζ parameter is bounded. Its values limit within the range of [-1,1]

(17) Kan et al. [21]:

Kan et al. [21] used the following plastic potential function:

$$\begin{cases} g(p',q,\bar{\theta},p_0) = \tilde{t}q + \frac{AM(\bar{\theta})p'}{A-1}\left((\frac{p'}{p_0})^{A-1} - 1\right) & for\ A \neq 1 \\ g(p',q,\bar{\theta},p_0) = \tilde{t}q + M(\bar{\theta})\,p'\ln\left(\frac{p'}{p_0}\right) & for\ A = 1 \end{cases} \qquad (47)$$

in which p_0 controls the size of the plastic potential surface, and *A* is a material parameter.

$\tilde{t}$ is a scalar whose sign controls the direction of plastic flow. It takes value of 1 or -1 depends on relative positions of the stress point, σ', on the yield surface and its corresponding image point, $\tilde{\sigma}'$, on the bounding surface.

$M(\bar{\theta})$ is calculated as a function of load angle, $\bar{\theta}$, as follows:

$$M(\bar{\theta}) = M\left(\frac{2\alpha^4}{1+\alpha^4-(1-\alpha^4)\sin 3\bar{\theta}}\right)^{\frac{1}{4}} \qquad (48)$$

Where load angle $\bar{\theta}$ is defined as:

$$\bar{\theta} = \frac{1}{3}sin^{-1}(-\frac{3\sqrt{3}}{2}\frac{J_3}{\sqrt{(J_2)^3}}) \qquad (49)$$

in which J_2 and J_3 are the second and thrid invariants of the deviatoric stress tensor. Note that the load angle ranges from $\bar{\theta}$=30 for triaxial compression to $\bar{\theta}$=-30 for triaxial extension.

a is determined by below relation:

$$\alpha = \frac{3-sin\varphi_{cs}}{3+sin\varphi_{cs}} \qquad (50)$$

M is the value of the critical state stress ratio under triaxial compression which is linked to the critical state friction angle as:

$$M = \frac{6sin\varphi_{cs}}{3-\varphi_{cs}} \qquad (51)$$

The direction of plastic flow in this model is calculted based on the following relations:

$$m_g = \frac{\frac{\partial g}{\partial\sigma'}}{\left\|\frac{\partial g}{\partial\sigma'}\right\|} \qquad (52)$$

$\partial g/\partial\sigma'$ is evaluated by applying the chain rule of differentiation as follows:

$$\frac{\partial g}{\partial\sigma'} = \frac{\partial g}{\partial p'}\frac{\partial p'}{\partial\sigma'} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial\sigma'} + \frac{\partial g}{\partial\bar{\theta}}\frac{\partial\bar{\theta}}{\partial\sigma'} \qquad (53)$$

(18) Gao et al. [11]:

The following fabric-dependent dilatancy equation was proposed by Gao et al. [11]:

$$d = \frac{d_1}{Mg(\theta)}\left[1+\frac{R}{Mg(\theta)}\right]\left[Mg(\theta)\exp\left(m_1\zeta_g\right) - R\right] \qquad (54)$$

whrer d_1 and m_1 are two model constants. ζ_g is defined as follows:

$$\zeta_g = \xi - e_A(A-1)$$

where e_A is a model parameter, $\xi=e-e_{cr}$ is the state parameter, *A* is a fabric anisotropy variable, *R* is the sress ratio tensor which is defined as: $R = \sqrt{3/2r_{ij}r_{ij}}$ in which $r_{ij} = (\sigma_{ij} - p\delta_{ij})/p = s_{ij}/p$, in which σ_{ij} is the stress tensor, $p=\sigma_{ij}/3$ is the mean normal stress, δ_{ij} is the Kronecker delta and S_{ij} is the deviator stress. $g(\theta)$ is defined based on the load angle θ as:

$$g(\theta) = \frac{\sqrt{(1+c^2)^2+4c(1-c^2)sin3\theta} - (1+c^2)}{2(1-c)sin3\theta} \qquad (55)$$

where $c=M_e/M_c$ is the ratio between the critical state stress ratio in triaxial extension M_e and that in triaxial compression M_c.

A is defined based on product of the fabric tensor F_{ij} and the loading direction tensor n_{ij} as:

$$A = F_{ij}n_{ij} \qquad (56)$$

where F_{ij} is a symmetric tensor whose norm $F = \sqrt{F_{ij}F_{ij}}$ is named as the degree of fabric.

(19) Qu and Huang [39]:

Similar to Li and Dafalias [29], dilatancy of sand was related to state parameter ξ and the current stress ratio $\eta=q/p'$ by Qu and Huang [39] as follows:

$$d = \frac{d_0}{M}\left[Mg(\theta)e^{m\xi} - \eta\right] \qquad (57)$$

where d_0 and *m* are material parameters and *M* is the critical state stress ratio under triaxial compression.

$g(\theta)$ is defined similar to the definition which was given in Gao et al. [11]. θ is the load angle which is defind similar to Kan et al. [21].

Mean effective stress p' and deviatoric stress *q* are calculated by $p' = I_1/3$ and $q = \sqrt{3J_2}$ in which I_1 is the first invariant of the effective stress tensor and J_2 is the second invariant of the deviatoric stress tensor.

The proposed stress-dilatancy relationship gives rise to the following plastic potential function:

$$g(p^{'},q,p_c) = q + Me^{m\xi}g(\theta)p^{'}ln\left(\frac{p^{'}}{p_c}\right) \tag{58}$$

(20) Jin et al. [20]:

Jin et al. [20] presented the following stress-dilatancy equation for cohesionless sands:

$$d = d_a(M_{PT} - \eta)\ exp\left[h_2\left(\frac{e}{e_{cr}} - 1\right)\right] \tag{59}$$

h_2 is a material parameter and d_a is a positive constant which controls the magnitude and evolution of dilatancy rate.

M_{PT} is the current phase transformation stress ratio, η is the current stress ratio, e is the current void ratio, and e_{cr} is the critical state void ratio.

The term (e/e_{cr}-1) is used to adjust dilation or contraction of loose and dense sands. For loose sands $e > e_{cr}$ and $\eta < M_{PT}$, now if $h_2 > 0$ then the exponential term increases the rateof contraction as d_a does. Also if $h_2 < 0$ then the exponential term decreases the rate of contraction. However as shearing continues towards the critical state, the effect of the exponential term on the stress dilatancy diminishes because the exponential term approaches zero. When stress state reaches the critical state, the exponential term totally disappears and $M_{PT}=\eta$ which guarantees zero dilatancy rate at the critical state.

(21) Sun et al. [53]:

Sun et al. [53] investigated the effect of particle breakage on the stress-dilatancy behavior of granular soils and proposed the following plastic flow rule with modification of Rowe's equation:

$$d = \frac{9(M - \eta) + 9\eta\chi}{9 + 3M - 2M\eta - \chi} \tag{60}$$

where

$$\chi = \frac{\gamma(6 + 4M)(3 - M)}{3(6 + M)} \tag{61}$$

in which γ is a material parameter which slightly depends on confining pressure. M is the critical state stress ration under triaxial compression. Therefore, x is a constant.

(22) Yin et al. [70]:

The following simple non-associated plastic potential function was suggested by Yin et al. [70] for very coarse granular materials:

$$g(p^{'},q) = \frac{q}{M_{PT}\,p^{'}} + \ln(p^{'}) \tag{62}$$

where $M_{PT} = 6sin\varphi_{PT}/(3 - sin\varphi_{PT})$ in which φ_{PT} is the phase transformation or characteristic angle which corresponds to transition from the contractive to dilative behavior.

(23) Sun and Xiao [52]:

The following plastic potential function was presented for granular soils by Sun and Xiao [52]:

$$g = (2p^{'} - p^{'}_o)^2 + \left(\frac{2q}{M}\right)^2 - P^{'2}_0 \tag{63}$$

where $p^{'}_0$ controls the size of the plastic potential surface, $p^{'}$ is the mean effective stress, and q is the deviatoric stress.

The following stress-dilatancy relation was also suggested for granular soils by Sun and Xiao [52]:

$$d = \frac{M^2 - (1 - \alpha_d/2)(\eta^2 + M^2)}{\eta^{2-\alpha_d}} \tag{64}$$

where

$$\alpha_d = \exp(\beta_d\xi) \tag{65}$$

in which β_d is a material parameter and ξ is the state parameter.

If the sand is sheared from a dense state where the state parameter is negative ($\xi < 0$), a_d would increase with shearing until the critical state is ultimately reached when $\xi=0$, $a_d=1$ and $d=0$. However, if the sand is initially at a loose state where the state parameter is positive ($\xi > 0$), a_d, would decrease until the critical state is reached when $\xi=0$, $a_d=1$, and $d=0$.

3.2 Cemented Sands

A summary of the following stress-dilatancy relationships and plastic potential functions of cohesive sands with important highlights is given in table 4 (see appendix section).

(1) Vermeer and De Borst [59]:

Rowe's stress-dilatancy equation in terms of mobilized dilation and friction angle takes the following form [28,59]:

$$sin\psi_m = \frac{sin\varphi_m - sin\varphi_{cv}}{1 - sin\varphi_m sin\varphi_{cv}} \tag{66}$$

where φ_{cv} is the constant volume or critical state friction angle. ψ_m is the mobilized dilatancy angle which is a function of plastic strain as the mobilized friction angle （φ_m） does. Thus, change in ψ_m can be predicted by change in φ_m.

The above equation was initially proposed by Vermeer and De Borst [59] for soils, rocks and concrete. However,

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this form of Rowe's equation can not describe the density or void ratio dependency during shearing. In order to address this shortcoming, a modification of the original Rowe's equation was proposed by Wan and Guo [60] as follows:

$$sin\psi_m = \frac{sin\varphi_m - (e/e_{cr})^{\omega_d}\, sin\varphi_{cv}}{1 - (e/e_{cr})^{\omega_d}\, sin\varphi_m sin\varphi_{cv}} \tag{67}$$

where e is the current void ratio, e_{cr} is the critical state void ratio, and ω_d is a material parameter. Parameter ω_d can also be linked to fabric tensor, plastic shear strain and the effect of intermediate principal stress measured by parameter $b = (\sigma_2 - \sigma_3)/(\sigma_1 - \sigma_3)$ [60].

(2) Kim and Lade [24]:

Based on variety of experimental observations, Kim and Lade [24] suggested the following equation for the plastic potential function of cohesive and non-cohesive frictional materials:

$$g(I_1, I_2, I_3) = \left(\psi_1 \frac{I_1^3}{I_3} - \frac{I_1^2}{I_2} + \psi_2\right)\left(\frac{I_1}{p_a}\right)^{\mu} \tag{68}$$

where ψ_1, ψ_2, and μ are the plastic potential parameters, p_a is the atmospheric pressure and I_1, I_2, and I_3 are the first, second, and third invariant of the stress tensor which are defined based on the three principal stresses (observed during triaxial tests) as follows [24]:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \tag{69}$$

$$I_2 = -\,(\sigma_1.\sigma_2 + \sigma_2.\sigma_3 + \sigma_3.\sigma_1) \tag{70}$$

$$I_3 = \sigma_1.\sigma_2.\sigma_3 \tag{71}$$

Reddy and Saxena [45], Lade and Kim [26], Kandasami et al. [22], Singh et al. [50], and Singh et al. [51] used the above plastic potential function in their constitutive model to simulate the plastic flow of cemented and uncemented geomaterials.

(3) Van den Hoek and Geilikman [58]:

Van den Hoek and Geilikman [58] presented the following plastic potential function for sandstone in analysis of sand production in petroleum wells:

$$g = \frac{1}{2}(\sigma_\vartheta^{'} - \sigma_r^{'}) - \frac{1}{2}(\sigma_\vartheta^{'} + \sigma_r^{'})sin\psi + constant \tag{72}$$

where ψ is the dilation angle, $\sigma_\vartheta^{'}$ is the tangential effective stress and $\sigma_r^{'}$ is the radial effective stress. Constant value has no impact on the model since only the derivative of the plastic potential function is influential in constitutive model.

(4) Arroyo et al. [3]:

Arroyo et al. [3] proposed the following stress-dilatancy equation for cemented sands:

$$d = m_g(M - \eta)\left(\frac{a_g M}{\eta}\right) \tag{73}$$

where m_g, a_g, and M are model parameters which control the shape of the plastic potential surface.

Note that the above stress-dilatancy relationship is similar to that presented by Lagioia et al. [27]. Buscarnera and Laverack [5] also used the similar flow rule as above in their constitutive model presented for porous rocks.

(5) Yu et al. [73]:

The following stress-dilatancy relationship was proposed by Yu et al. [73] for cemented soils:

$$d = \frac{9(M - \eta) + 6\dfrac{coh}{p^{'}}\sqrt{(2M + 3)(-M + 3)}}{9 + 3M - 2M\eta + 4\dfrac{coh}{p^{'}}\sqrt{(2M + 3)(-M + 3)}} \tag{74}$$

where *coh* is the interparticle cohesion. Original Rowe's flow rule can be recovered by setting *coh=0*. In this model, cohesion was assumed to degrade with the total plastic strain increment as follows:

$$dcoh = coh \exp\left(-\varpi\,.\sqrt{(\dot{\varepsilon}_q^p)^2 + (\dot{\varepsilon}_p^p)^2}\right) \tag{75}$$

where ϖ determines rate of cohesion degradation.

The plastic potential function corresponds to the above stress-dilatancy equation is as follows [73]:

$$\begin{aligned} g = {} & 3M\, ln\left(\frac{p^{'} + p_t^{'} - k_g}{\varphi_g}\right) + (2M + 3)\, ln\left(\frac{2(q - h_g)}{p^{'} + p_t^{'} - k_g} + 3\right) + \\ & (M - 3) ln\left(3 - \frac{q - h_g}{p^{'} + p_t^{'} - k_g}\right) \end{aligned} \tag{76}$$

$$k_g = \frac{\sqrt{(3 + 2M)(3 - M)}\,(36 - 12M)coh}{18M^2 - 27M - 81} \tag{77}$$

$$h_g = \frac{\sqrt{(3 + 2M)(3 - M)}\,(-54 + 18M)coh}{18M^2 - 27M - 81} \tag{78}$$

Rahimi et al. [41], Rahimi et al. [42] and Rahimi et al. [44] used the same stress-dilatancy relationship as that proposed by Yu at al. [73] by replacement of constant critical state stress ratio with variable phase transformation stress ratio.

(6) Wang and Leung [63]:

Wang and Leung [63] presented the following stress-dilatancy relationship for cemented sands in triaxial condition (i.e. an axisymmetric condition):

$$q\dot{\varepsilon}_q^p + p^{'}\dot{\varepsilon}_P^p = Mp^{'}\dot{\varepsilon}_q^p + \Delta W_{bond} \tag{79}$$

or

$$\frac{q}{p^{'}} + \frac{\dot{\varepsilon}_P^p}{\dot{\varepsilon}_q^p} = M + \frac{\Delta W_{bond}}{p^{'}\dot{\varepsilon}_q^p} \qquad or \qquad d = M - \eta + \frac{\Delta W_{bond}}{p^{'}\dot{\varepsilon}_q^p} \tag{80}$$

where ΔW_{bond} is the total energy which is dissipated due to breakage or destruction of the cementing bond.

Ajorloo et al. [1] also suggested the same stress-dilatancy relation for cemented sands. However, neither Wang and Leung [63] nor Ajorloo et al. [1] did not put forward any formulas for .ΔW_{bond} Suggestion of any relationship for ΔW_{bond} will be considered as a contribution for the stress-dilatancy relation of cemented sands.

(7) Zhang and Salgado [74]:

Since original Rowe's stress-dilatancy relationship does not produce correct volumetric response for cemented materials [73, 74, 40, 41], a modification of Rowe's equation was proposed by Zhang and Salgado [74] as follows to address this deficiency:

$$\frac{\sigma_1^{'}}{\sigma_3^{'}} = K\left(1 - \frac{\dot{\varepsilon}_V}{\dot{\varepsilon}_1}\right) + \frac{2coh}{\sigma_3^{'}}K\sqrt{1 - \frac{\dot{\varepsilon}_V}{\dot{\varepsilon}_1}} \tag{81}$$

where K is a parameter which depends on the failure friction angle, $\sigma_1^{'}$ and $\sigma_3^{'}$ are the major and minor principal effective stresses, respectively, $\dot{\varepsilon}_V$ and $\dot{\varepsilon}_1$ are the volumetric and major principal strain increment.

The above equation can be written in terms of the mean effective stress $p^{'}$, the deviator stress q, and the critical state stress ratio under compression M as follows:

$$d = \frac{9(M - \eta) - 3m_c}{9 + 3M - 2\eta M + m_c} \tag{82}$$

where m_c is related to the cohesion as follows:

$$m_c = \frac{6(-M+3)\left(coh/p^{'}\right)^2}{-\eta+3} - \frac{2coh(-M+3)}{p^{'}}\sqrt{\left(\frac{3coh/p^{'}}{-\eta+3}\right)^2 + \frac{2\eta+3}{-\eta+3}} \tag{83}$$

Where coh is the interparticle cohesion.

Note that Zhang and Salgado [74] and Yu et al. [72,73]'s stress-dilatancy relationships have been mentioned in Porcino and Marciano [38] as well.

(8) Gao and Zhao [10]:

Gao and Zhao [10] presented the following flow rule for cemented sands:

$$d = \frac{d_1}{exp\left(\int \langle dL \rangle\right)}\left(M_{PT}d_Cd_F - H\right) \tag{84}$$

where d_1 is a positive model parameter, M_{PT} is the phase transformation stress ratio, H is hardening parameter, DL is a loading index and $\langle x \rangle$ denotes the Macauley bracket with $\langle x \rangle$=0 when $x \le 0$ and $\langle x \rangle = x$ when $x > 0$. The role of the denominator in above equation is to control the volume change, especially when the strain level is high. As the sample is sheared to the critical state, the increment of the plastic deviatoric strain is unlimited. At this state, the denominator term will reach infinity making the value of the dilatancy rate d zero. The two parameters d_C and d_F are used to characterize the bonding and anisotropic effects, respectively, as follows:

$$d_C = exp\left(-c_0\sqrt{\frac{\sigma_0}{p_r}}\right) \tag{85}$$

$$d_F = exp(k(A+1)) \tag{86}$$

where C_0 and k are positive model constants, σ_0 denotes the current triaxial tensile strengthof the material, p_r is a reference pressure and A is an anisotropic variable dependent on fabric tensor whose definition has given in Gao and Zhao [10].

Above equations imply that as the value of tensile strength increases, d_C decreases which leads to reduction in value of the dilatancy rate d. The material response is less contractive and hence the liquefaction resistant increases. Also d_F increases with A (the major principal stress direction becomes closer to the direction of deposition) which in turn results in larger value of the dilatancy rate d and hence more contractive response and less resistant to the liquefaction [10].

(9) Weng [65]:

In his generalized plasticity model for sandstone, Weng [65] used Nova and Wood [36]'s stress-dilatancy relationship which has been proposed for cohesionless sands. This, however, is a questionable choice because cemented sands show more dilative behavior than uncemented sands [40, 41]. This can not be reproduced simply by using the flow rule developed for cohesionless sands.

(10) Rios et al. [46]:

Rios et al. [46] applied Rowe's stress-dilatancy relation (equation 10) in their constitutive model developed for modelling bonded soils. This is controversial. As men-

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tioned already, the stress-dilatancy relationships proposed for cohesionless sands do not work for cemented or bonded sands because volumetric behavior of uncemented sands is different than that of cemented sands [40, 41].

4. Conclusion

Some important stress-dilatancy relationships and plastic potential functions proposed for uncemented and cemented sands were reviewed in this article. The paper suggests that unlike uncemented sands for which intensive research has been performed by different researchers, there are a few researches on the stress-dilatancy relationships of cemented sands. To fill this gap, considerable effort is needed. Therefore, it is recommended that this area (stress-dilatancy relationship of cemented sands) is chosen as a potential research title for future studies. Modification of Rowe's equation can be a step forward to account for deficiencies of the previous stress-dilatancy relationships proposed for cemented sands.

Conflicts of interest

The corresponding author confirms that there is no conflict of interest.

Appendix

Summary of aforementioned flow rule and plastic potential equations for uncemented and cemented sands

Table 3. Summary of proposed stress-dilatancy and plastic potential relationships for uncemented sands

Proposed stress-dilatancy relationship or plastic potential function	Methodology/Application/Validation (Calibration)
Nova and Wood [35]	-proposed on the basis of experimental works on sand in plane strain & triaxial conditions. -when $\mu=1$, the equation reduces to that of the Cam Clay model. -verified against constant p' and constant q laboratory tests with great matches.
Nova and Wood [36]	-One of the oldest and most popular stress-dilatancy relationships. -This relationship has been employed by many researchers like Pastor et al. [37], Haeri and Hamidi [13], Hamidi and Yarbakhti [14] and Kong et al. [25].
Jefferies [17]	-proposed in his critical state constitutive model. -when $N=0$, the equation reduces to the familiar Cam Clay model. -the influence of void ratio and mean effective stress on sand response is captured well but the influence of fabric is neglected.
Lagioia et al.[27]	-proposed such that it predicts zero dilatancy rate in the critical state and infinite dilatancy rate in zero stress ratio. -verified against drained triaxial tests with excellent matches.
Yu [71]	-proposed for both sand & clay based on Rowe's stress-dilatancy relationship. -verified against drained & undrained triaxial compression tests of sand & clay with very good agreements.
Schanz et al. [15]	-proposed based on concept of mobilized dilatancy angle, mobilized friction angle and critical state friction angle. -the mobilized friction angle is estimated by shape of a Mohr-Coulomb failure envelope. -calibrated for drained & undrained behavior of Hostun loos sand under oedometer & triaxial tests.
Wan and Guo [61]	-a fabric dependent stress-dilatancy equation. -successfully calibrated against undrained & drained behavior of Ottawa & Toyoura sands.
McDowell [31]	-the flow rule is non-associated. Hence, the critical state point does not happen at the top of yield surface. -fulfills the necessary conditions of zero shear strain under isotropic conditions and infinite shear strain with zero volumetric strain at the critical state. This approach is identical to that suggested by Lagioia et al. [27] -performance of the model was discussed, however, it was not verified against experimental observations.
Russell and Khalili [48]	-A state parameter dependent stress-dilatancy equation within critical state framework. -Comparison of predicted volumetric behavior with experimental results (under isotropic & oedometric compression, and drained & undrained triaxial tests) suggests a great credit for the proposed equation.
Dafalias and Manzari [33]	-a fabric dependent stress-dilatancy equation based on difference between the current stress ratio and the phase transformation stress ratio. -The model predicts a phase transformation stress ratio equal to the current stress ratio at the critical state which results in zero dilatancy rate (permanent steady state response). -calibrated against drained & undrained monotonic & cyclic compression tests. Predicted volumetric responses were in very good harmony with experimentally measured behaviors.
Khalili et al. [23]	-similar to equation of Russell and Khalili [48]. -validated excellently against drained & undrained compression monotonic & cyclic tests on loose & dense samples of Hostun sand.
Imam et al. [16]	-similar to equation of Dafalias and Manzari [33], however, it is not fabric dependent. -proposed for both triaxial compression and triaxial extension. -volumetric response was validated against many drained & undrained triaxial compression & extension tests on Ottawa and Toyoura sands under monotonic loading with appreciable matches.
Ling and Yang [30]	-a combination of relationships proposed by Nova and Wood [36] and Li and Dafalias [29]. -a variable stress ratio (M_f) is used in this equation rather than the constant critical state stress ratio (M). -wondeful performance of the equation was proved by calibration of drained & undrained volumetric behavior of Toyoura & Nevada & Fuji River sands under monotonic & cyclic loading.

Sasiharan [49]	-a fabric dependent equation. -Similar to Jefferies's approach, Sasiharan considered the resulting plastic potential function, obtained by integration of the flow rule, identical to the yield function. -calibrated against drained & undrained triaxial compression tests under monotonic & cyclic loading with good matches.
Tasiopoulou and Gerolymos [55]	-a hardening parameter & phase transformation stress ratio dependent equation. -performance of the model was examined but it was not validated against experimental observations.
Kan et al. [21]	-a relatively complex equation for plastic potential function. -calibrated successfully against drained & undrained cyclic tests on loose & dense samples of Toyoura & Fuji River sands.
Gao et al. [11]	-a relatively complex fabric-dependent dilatancy equation. -calibrated against drained & undrained behavior of Toyoura & Fraser River sands under different loading conditions.
Qu and Huang [39]	-a state parameter dependent dilatancy equation. -was not verified based on experimental observations.
Jin et al. [20]	-a void ratio & phase transformation stress ratio dependent equation. -calibrated based on drained & undrained triaxial compression tests of Nevada & Toyoura sands under monotonic & cyclic loading.
Sun et al. [53]	-a modified version of Rowe's equation which considers the effect of particle breakage. -was calibrated against plastic flow of Yixing rockfill, Kish Island & Cambria sands
Yin et al. [70]	-a simple non-associated plastic potential function suggested for very coarse granular materials. -calibrated based on drained triaxial tests on rockfill materials with different grain sizes.
Sun and Xiao [52]	-a state parameter dependent dilatancy equation for granular soils subjected to monotonic triaxial compression. -calibrated successfully against drained & undrained triaxial compression tests on various granular soils.

Table 4. Summary of proposed stress-dilatancy and plastic potential relationships for cemented sands

Proposed stress-dilatancy relationship or plastic potential function	Methodology/Application/Validation (Calibration)
Vermeer and De Borst [59]	-Rowe's stress-dilatancy equation in terms of mobilized dilation and friction angles. -proposed for soils, rocks and concrete. -a modified version that takes into account the density or void ratio dependency during shearing was proposed by Wan and Guo [60].
Kim and Lade [24]	-a plastic potential function for cohesive and non-cohesive frictional materials proposed based on many experimental observations.
Van den Hoek and Geilikman [58]	-a plastic potential function proposed for analysis of sand production in petroleum wells. -sand production analysis was performed, however, volumetric behavior of sandstone was not calibrated against physical tests.
Arroyo et al. [3]	-flow rule similar to that of Lagioia et al. [27]. -no strong validation against physical tests.
Yu et al. [73]	-a modified version of Rowe's flow rule proposed for modelling bonded geomaterials. -satisfactory agreement of volumetric behavior for numerical and physical tests. Also no validation was performed for undrained tests.
Wang and Leung [63]	-proposed for cemented sands under triaxial condition (i.e. an axisymmetric condition): -no calibration was done to validate the proposed flow rule.
Zhang and Salgado (2010)	- a modification of Rowe's equation -no verification against experimental observations.
Gao and Zhao [10]	-a fabric anisotropy dependent stress-dilatancy relationship. -successful calibration of volumetric behavior of cemented Ottawa sand in drained triaxial compression & Toyoura sand in drained true triaxial tests. -no validation against undrained tests.
Weng [65]	-identical to Nova and Wood [36]'s stress-dilatancy equation. -arguable choice because cohesive sands show more dilative response than uncemented sands.
Rios et al. [46]	-identical to Rowe's stress-dilatancy equation. -controversial since volumetric behavior of uncemented sands is different than that of cohesive sands.

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