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**REVIEW**

# Remark On Optimal Homotopy Method: Application Towards Nano-Fluid Flow Narrating Differential Equations

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ABSTRACT

The short communication is devoted to validate the reliability and convergence of Optimal Homotopy Analysis Method (O-HAM). Owing the importance of present validation of O-HAM one can implement this method towards nanofluid flow narrating differential equations at larger scale for better description. To be more specific, the fractional order differential equation due to vertically moving non-spherical nano particle in a purely viscous liquid and an advection PDE is take into account. The corresponding homotopy for both cases are constructed and solutions are proposed by means of O-HAM. The obtained values are compared with numerical benchmarks. We observed an excellent match which confirms the O-HAM conjecture. Therefore, it can be directed that the utilization of O-HAM towards nanofluid flow regime may provide relief against some non-attempted problems.

## 1. Introduction

Fractional order differential equations have wide range of application in nature. Many scientists and mathematicians are working on fractional calculus [1-2] now a days. It is important to get convergent solution of any physical phenomenon that's why researchers are working to invent or modify the algorithms to solve the problems. An Optimal Homotopy Analysis Method[3,4,10] is one of the modified algorithm involving convergence control parameters to speed up the convergence. In current article remark application of Optimal Homotopy Analysis Method (OHAM) towards convergent solutions of fractional order differential equations. To show the efficiency of the method on the physical problems of nature two examples are given representing the fluid flow. For better understating firstly we need to introduce the basics involved in Optimal Homotopy Analysis (OHAM). Consider the following boundary value problem

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$$A(u(z,t)) + f(z,t) = 0,\ z \in \Omega\ , \tag{1}$$

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0,\quad z \in \Gamma\ , \tag{2}$$

where, $u(z,t)$ is a function while $z$ and $t$ denote spatial and temporal independent variables respectively, $B$ is the boundary and $A$ is a differential operator, the domain $\Omega$ has boundary $\Gamma$ and $f(z,t)$ is an analytic known function. $A$ can be written as:

$$A = N + L\ , \tag{3}$$

where, $L=D_*^{\alpha}(.)$ is linear (fractional order) and $N$ is nonlinear operator. $A$ Homotopy can be constructed by means of OHAM as follows:

$$\phi(z,t;p) : \Omega \times [0,1] \to R, \tag{4}$$

which satisfies

$$H\left(\phi(z,t;p), p\right) = (1-p)\{L(\phi(z,t;p)) + f(z,t)\} - H(p)\{A(\phi(z,t;p)) + f(z,t)\} = 0, \tag{5}$$

here, $p$ is an embedding parameter where $p \in [0,1]$ and $H(p)$represents an auxiliary function which is nonzero. Further, we have following possibilities

$H(p)$ for $p \neq 0$.

For $p = 0 : H(\varphi(z,t;0),0) = L(\varphi(z,t;0)) + f(z,t) = 0,$ (6)

for $p = 1 : H(\phi(z,t;1),1) = H(1)\{(A\phi(z,t;1)) + f(z,t)\} = 0.$ (7)

As $p$ varies from 0 to 1, and the $\phi(z,t;p)$ varies from $\phi(z,t;0) = u_0(z,t)$ to $\phi(z,t;1) = u(z,t)$ respectively, where $u_0(z,t)$ can be obtained from Eq. (5) and Eq. (2). In addition,

$$L(\phi(z,t;0)) + f(z,t) = 0, \qquad B\left(u_0, \frac{\partial u_0}{\partial t}\right) = 0. \tag{8}$$

Taylor's series of $\phi(z,t;p,c_i)$ can be expanded about $p$ as follows:

$$H(p) = pc_1 + p^2c_2 + p^3c_3 + \ldots$$

$$\phi(z,t;p,c_i) = u_0(z,t) + \sum_{k=1}^{\infty} u_k(z,t;c_i)p^k, \qquad i = 1,2,3,.... \tag{9}$$

combining Eq. (9) and Eq. (5) and equating the coefficient of like powers of $p$, we obtain the zeroth-order problem, given by Eq. (8). Similarly, the first and second order problems are obtained as follows:

$$L(u_1(z,t;0)) = C_1N_0(u_0(z,t)), \quad B\left(u_1, \frac{\partial u_1}{\partial t}\right) = 0, \tag{10}$$

And

$$L(u_2(z,t;0)) - L(u_1(z,t;0)) = C_2N_0(u_0(z,t)) + C_1\left[L(u_1(z,t)) + N_1(u_0(z,t), u_1(z,t))\right], \tag{11}$$

$$B\left(u_2, \frac{\partial u_2}{\partial t}\right) = 0. \tag{12}$$

The general governing equations for $u_k(z,t)$ are given as follows:

$$L(u_k(z,t)) = L(u_{k-1}(z,t)) + C_kN_0(u_0(z,t)) + \sum_{i=1}^{k-1} C_i\left[L(u_{k-i}(z,t)) + N_{k-i}(u_0(z,t), u_1(z,t), \ldots u_{k-i}(z,t))\right],$$
$$k = 2,3,\ldots B\left(u_k, \frac{\partial u_k}{\partial t}\right) = 0, \tag{13}$$

where, $N_{k-i}(u_0(z,t), u_1(z,t), \ldots u_{k-i}(z,t))$ is the coefficient of $p^{k-i}$ in the expansion of $N(\phi(z,t;p))$ about the embedding parameter $p$ and

$$N(\phi(z,t;p,C_i)) = N_0(u_0(z,t)) + \sum_{k\geq 1} N_k(u_0, u_{1,\ldots,u_k})p^k. \tag{14}$$

The convergence of Eq. (9) depends upon the constants $C_1$, $C_2$, $C_3$..., We take convergent at $P$=1, therefore

$$\tilde{u}(z,t;C_i) = u_0(z,t) + \sum_{k\geq 1} u_k(z,t;C_i). \tag{15}$$

The residual is obtained by invoking Eq. (15) in Eq. (1), we have

$$R(z,t;C_i) = L(\tilde{u}(z,t;C_i)) + f(z,t) + N(\tilde{u}(z,t;C_i)). \tag{16}$$

It is important to note that the $R(z,t;C_i)$=0 if solution is n . It is nonzero in case of nonlinear problems. To find the constants $C_1$, $C_2$, $C_3$..., one can use any of the methods available in the literature like Ritz, least squares, Galirkin's and collocation method to mention just a few. For instance the method of least squares is initiated:

$$J(C_i) = \int_0^t \int_\Omega R^2(z,t;C_i)\,dz\,dt, \tag{17}$$

where, $R$ is the residual, so

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = \ldots = \frac{\partial J}{\partial C_m} = 0, \tag{18}$$

the solution of above system of equations will yield an auxiliary parameters. The validation subject to OHAM is elaborated case-wise

## 2. Non-spherical particle

The vertically falling non spherical particle in purely viscous fluid yield the fractional order differential equation as follows:

$$D_{*_t}{}^{\alpha}u(t)-\frac{w}{q}+\frac{r}{q}u(t)+\frac{s}{q}u^2(t)=0,\ u(0)=0,\ 0\le\alpha\le 1. \tag{19}$$

The corresponding homotopy equation can be constructed as

$$(1-p)\left\{\frac{d^{\alpha}u}{dt^{\alpha}}-\frac{w}{q}\right\}-H(p,c_i)\left\{\frac{d^{\alpha}u}{dt^{\alpha}}-\frac{w}{q}+\frac{r}{q}u+\frac{s}{q}u^2\right\}=0. \tag{20}$$

Now consider the following relations

$$u(t)=u_0(t)+pu_1(t)+p^2u_2(t)\ \ and\ \ H(p,c_i)=pc_1+p^2c_2, \tag{21}$$

By incorporating Eq. (21) into Eq. (20). Further, by regrouping we have following zeroth, first and second order problems

$$\frac{d^{\alpha}u_0}{dt^{\alpha}}-\frac{w}{q}=0,\ u_0(0)=0. \tag{22}$$

$$\frac{d^{\alpha}u_1}{dt^{\alpha}}-\frac{d^{\alpha}u_0}{dt^{\alpha}}+\frac{w}{q}-\left(c_1\frac{d^{\alpha}u_0}{dt^{\alpha}}-c_1\frac{w}{q}+c_1\frac{r}{q}u_0+c_1\frac{s}{q}u_0{}^2\right)=0,\ u_1(0)=0. \tag{23}$$

$$\frac{d^{\alpha}u_2}{dt^{\alpha}}-\frac{d^{\alpha}u_1}{dt^{\alpha}}-\left(\begin{array}{l} c_1\dfrac{d^{\alpha}u_1}{dt^{\alpha}}+c_1\dfrac{r}{q}u_1+2c_1\dfrac{s}{q}u_0u_1 \\ +c_2\dfrac{d^{\alpha}u_0}{dt^{\alpha}}+c_2\dfrac{w}{q}+\dfrac{r}{q}u_0+c_2\dfrac{s}{q}u_0{}^2 \end{array}\right)=0,\ u_2(0)=0. \tag{24}$$

By solving Eqs. (22)-(24) we have :

$$u_0=\frac{t^{\alpha}w}{q\Gamma(\alpha+1)}, \tag{25}$$

$$u_1=\left(\frac{(1+c_1)t^{\alpha}w}{q\Gamma(\alpha+1)}+\frac{t^{\alpha}w\left(q\left(-(1+c_1)q+\dfrac{t^{\alpha}r\sqrt{\pi}4^{-\alpha}c_1}{\Gamma(\alpha+\frac{1}{2})}\right)\Gamma(\alpha+1)+\dfrac{c_1st^{2\alpha}w\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)}{q^3\Gamma(\alpha+1)^2}\right). \tag{26}$$

$$u_2=\frac{c_2t^{\alpha}w}{q\Gamma(\alpha+1)}+(1+c_1)\left(\begin{array}{l}\dfrac{(1+c_1)t^{\alpha}w}{q\Gamma(\alpha+1)}+ \\ \dfrac{t^{\alpha}w\left(q\left(-(1+c_1)q+\dfrac{t^{\alpha}r\sqrt{\pi}4^{-\alpha}c_1}{\Gamma(\alpha+\frac{1}{2})}\right)\Gamma(\alpha+1)+\dfrac{c_1st^{2\alpha}w\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)}{q^3\Gamma(\alpha+1)^2}\end{array}\right)+ \tag{27}$$

$$\frac{\left(t^{\alpha}w\left(2^{2\alpha+1}c_1{}^2s^2t^{4\alpha}w^2\Gamma(\alpha+\frac{1}{2})\Gamma(2\alpha+1)\Gamma(4\alpha+1)^2+...\right)\right)}{\sqrt{\pi}q^5\Gamma(\alpha+1)^2\Gamma(2\alpha+1)\Gamma(3\alpha+1)\Gamma(4\alpha+1)\Gamma(5\alpha+1)}.$$



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By using Eq. (15) the complete solution can be obtained as:

$$u(t)=\frac{t^{\alpha}w}{q\Gamma(\alpha+1)}+\left(\frac{(1+c_1)t^{\alpha}w}{q\Gamma(\alpha+1)}+\frac{t^{\alpha}w\left(q\left(-(1+c_1)q+\frac{t^{\alpha}r\sqrt{\pi}\,4^{-\alpha}c_1}{\Gamma(\alpha+\frac{1}{2})}\right)\Gamma(\alpha+1)+\frac{c_1 s t^{2\alpha}w\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)}{q^{3}\Gamma(\alpha+1)^{2}}\right)$$

$$+\frac{c_2 t^{\alpha}w}{q\Gamma(\alpha+1)}+(1+c_1)\left(\frac{(1+c_1)t^{\alpha}w}{q\Gamma(\alpha+1)}+\frac{t^{\alpha}w\left(q\left(-(1+c_1)q+\frac{t^{\alpha}r\sqrt{\pi}\,4^{-\alpha}c_1}{\Gamma(\alpha+\frac{1}{2})}\right)\Gamma(\alpha+1)+\frac{c_1 s t^{2\alpha}w\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\right)}{q^{3}\Gamma(\alpha+1)^{2}}\right)+$$

$$\frac{\left(t^{\alpha}w\left(2^{2\alpha+1}c_1^{\,2}s^{2}t^{4\alpha}w^{2}\Gamma(\alpha+\frac{1}{2})\Gamma(2\alpha+1)\Gamma(4\alpha+1)^{2}+\ldots\right)\right)}{\sqrt{\pi}\,q^{5}\Gamma(\alpha+1)^{2}\Gamma(2\alpha+1)\Gamma(3\alpha+1)\Gamma(4\alpha+1)\Gamma(5\alpha+1)}+\ldots \tag{28}$$

## 3. Advection-PDE

The Advection partial differential equation with fractional order is written as:

$$D_{*t}{}^{\alpha}u(\chi,t)-\chi-\chi t^{2}+u(\chi,t)\frac{\partial}{\partial\chi}u(\chi,t)=0,\ u(\chi,0)=0,\ 0\le\alpha\le 1. \tag{29}$$

Using definition of homotopy, we have

$$(1-p)\left\{\frac{\partial^{\alpha}u(\chi,t)}{\partial t^{\alpha}}-\chi-\chi t^{2}\right\}-H(p,c_i)\left\{\frac{\partial^{\alpha}u(\chi,t)}{\partial t^{\alpha}}-\chi-\chi t^{2}+u(\chi,t)\frac{\partial}{\partial\chi}u(\chi,t)\right\}=0. \tag{30}$$

Now consider the following

$$u(t)=u_0(t)+pu_1(t)+p^{2}u_2(t)\ \text{ and }\ H(p,c_i)=pc_1+p^{2}c_2, \tag{31}$$

by substituting Eq. (31) into Eq. (30) and regrouping one has zeroth, first and second order problem as follows:

$$\frac{\partial^{\alpha}u_0(\chi,t)}{\partial t^{\alpha}}-\chi-\chi t^{2}=0,\ u_0(\chi,0)=0. \tag{32}$$

$$\frac{\partial^{\alpha}u_1(\chi,t)}{\partial t^{\alpha}}=(1+c_1)\frac{\partial^{\alpha}u_0(\chi,t)}{\partial t^{\alpha}}-(\chi+\chi t^{2})(1+c_1)+c_1u_0(\chi,t)\frac{\partial}{\partial x}u_0(\chi,t),\ u_1(\chi,0)=0, \tag{33}$$

$$\begin{aligned}\frac{\partial^{\alpha}u_2(\chi,t)}{\partial t^{\alpha}}=c_2\frac{\partial^{\alpha}u_0(\chi,t)}{\partial t^{\alpha}}+(1+c_1)\frac{\partial^{\alpha}u_1(\chi,t)}{\partial t^{\alpha}}-c_2(\chi+\chi t^{2})+c_2u_0(\chi,t)\frac{\partial}{\partial\chi}u_0(\chi,t)+\\ c_1u_1(\chi,t)\frac{\partial}{\partial\chi}u_0(\chi,t)+c_2u_0(\chi,t)\frac{\partial}{\partial\chi}u_1(\chi,t),\ u_2(\chi,0)=0.\end{aligned} \tag{34}$$

By solving (32)-(34) one can obtain

$$u_0 = \frac{t^{\alpha}\chi\left(2t^2+(1+\alpha)(2+\alpha)\right)}{\Gamma(\alpha+3)}, \tag{35}$$

$$u_1 = t^{\alpha}\chi\left(\frac{(1+c_1)\left(2+2t^2+3\alpha+\alpha^2\right)}{\Gamma(\alpha+3)}+\frac{1}{\Gamma(\alpha)}\left(\begin{array}{l}-\frac{1}{\alpha}-\frac{c_1}{\alpha}-\frac{2t^2}{2\alpha+3\alpha^2+\alpha^3}\\ -\frac{2c_1t^2}{2\alpha+3\alpha^2+\alpha^3}+\frac{4c_1t^{2\alpha}\Gamma(2\alpha)}{\Gamma(\alpha+3)\Gamma(1+3\alpha)}+\ldots\end{array}\right)\right), \tag{36}$$

$$u_2 = \chi t^{\alpha}\left(\begin{array}{l}\frac{c_2}{\Gamma(\alpha+1)}+\frac{2c_2t^2}{\Gamma(\alpha+3)}+\frac{c_2\left(2+3^2\alpha+\alpha^2+2t\right)}{\Gamma(\alpha+3)}\\ +\frac{c_1(1+c_1)(1+\alpha)^2(2+\alpha)^2t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(\alpha+3)^2\Gamma(1+3\alpha)}+\ldots\end{array}\right), \tag{37}$$

By using (13) one has the solution as:

$$u = t^{\alpha}\chi\left(\begin{array}{l}-\frac{c_2}{\Gamma(\alpha+1)}-\frac{2c_2t^2}{\Gamma(\alpha+3)}+\frac{c_2\left(2+3\alpha+\alpha^2+2t^2\right)}{\Gamma(\alpha+3)}+\frac{(1+c_1)\left(2+3\alpha+\alpha^2+2t^2\right)}{\Gamma(\alpha+3)}\\ +\frac{c_2\left(2+3\alpha+\alpha^2+2t^2\right)}{\Gamma(\alpha+3)}\end{array}\right)+\ldots,. \tag{38}$$

## 4. Graphical outcomes

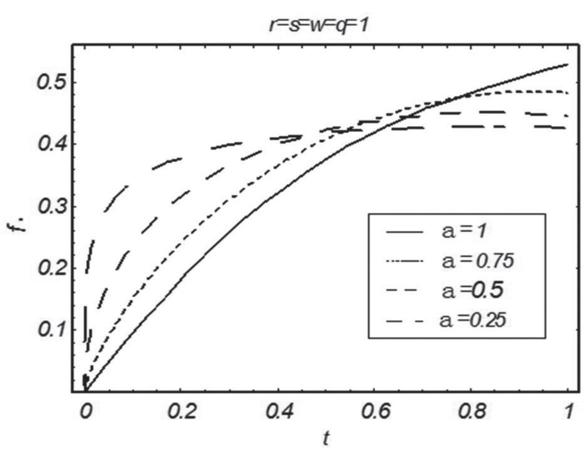


**Figure 1.** Velocity outcomes for Case-1

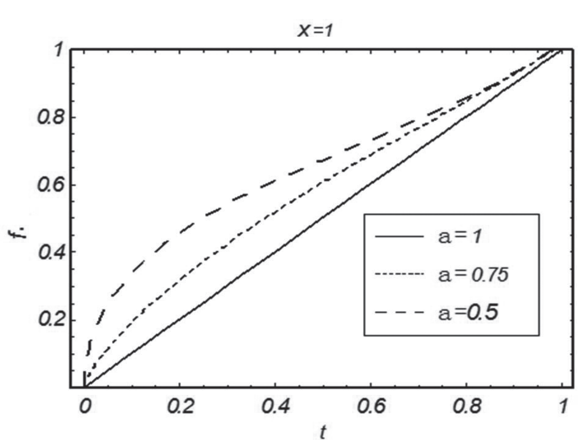


**Figure 2.** Velocity outcomes for Case-2

## 5. Table outcomes

**Table 1.** Numerical values and error estimation for

| $t$ | Numerical values | Error(Numerical-OHAM) |
|---|---|---|
| 0.1 | 0.0956117 | 0.000749743 |
| 0.2 | 0.18159 | 0.00247686 |
| 0.3 | 0.257138 | 0.00444689 |
| 0.4 | 0.321877 | 0.00587028 |
| 0.5 | 0.375914 | 0.00610734 |
| 0.6 | 0.419892 | 0.00486385 |
| 0.7 | 0.45506 | 0.00236298 |
| 0.8 | 0.483326 | -0.00051165 |
| 0.9 | 0.50732 | -0.0021004 |
| 1.0 | 0.530455 | 0.000125035 |

**Table 2.** Numerical values and error estimation for  and

| $t$ | Numerical values | Error (OHAM-VIM) |
|---|---|---|
| 0.1 | 0.100027 | 0.0000273273 |
| 0.2 | 0.200206 | 0.000205678 |
| 0.3 | 0.300624 | 0.000623707 |
| 0.4 | 0.401257 | 0.00125678 |
| 0.5 | 0.501942 | 0.00194152 |
| 0.6 | 0.602395 | 0.00239506 |
| 0.7 | 0.702307 | 0.00230681 |
| 0.8 | 0.801542 | 0.0015425 |
| 0.9 | 0.900517 | 0.000517285 |
| 1.0 | 1.00081340 | 0.00081724733 |



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## 6. Analysis

In this analysis the reliability and efficiency of Optimal Homotopy Analysis Method (O-HAM) is supported by considering differential system subject to non-spherical nano particle having motion vertically downward and PDE representing advection involvement. The outcomes are offered by way of graphical trends and tabular values. The Figs. 1-2 and Tables 1-2 are provided in this direction. To be more specific, Fig. 1 represents the velocity profiles for different fractional order differential equations have gradual change which ultimately gets closer to velocity profile for $\alpha$=1. $\alpha$=0.75 has convergence control parameters $c_1$ = -0.1229066, $c_2$ =-0.08944891, $\alpha$=0.5 has $c_1$ =-0.0364355, $c_2$ =-0.13012082 and $\alpha$=0.25 has $c_1$ = 0.0128925, $c_2$ =-0.1637184. From Fig. 2 one can see that towards advection partial differential equation the velocity profiles for different fractional order differential equations have gradual change which ultimately get closer to velocity profile for $\alpha$=1. $\alpha$=0.75. has convergence control parameters $c_1$ = -0.1229066, $c_2$ =-0.08944891and $\alpha$=0.5 has $c_1$ = -0.723306604, $c_2$ = 0.2237609808. Table1 offered the numerical values and error estimation for the velocity of non-spherical particle falling vertically down in the fluid obtained by Optimal Homotopy Analysis Method (O-HAM). The convergence control parameters for Table 1 are $c_1$ = -0.61310285 and $c_2$ = 0.00034590144. Error estimation shows that solution obtained by Optimal Homotopy Analysis Method is in good agreement with exact solution reported in Ref. [11]. Further, the Table 2 represents the numerical values and error estimation for the velocity profile of advection partial differential equation obtained by Optimal Homotopy Analysis Method. The convergence control parameters for Table 2 are $c_1$ = -1.0457108 and $c_2$ = 0.08153534. An error estimation shows that the solution obtained by Optimal Homotopy Analysis Method (O-HAM) is in good agreement with the solution obtained by Variational Iterational Method (VIM) and offered in Ref. [12]. It is trusted that one can implement such method on nanofluid flow narrating differential equation for better solution description.

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