Whether Achilles Could Catch Up the Turtle

Zhantao Yang*

The Gilbert School, No.200 Williams Avenue, Winsted, CT 06098, USA

Abstract: Can a person chase up a turtle? This might sound like a very stupid question. However, this question had been disturbing mathematicians for many centuries. It is possible to solve the paradox of Achilles and the turtle with only high school knowledge. This is the beauty of calculus, the concept that mathematicians struggled to invent and improve. In more than 2000 years, mathematicians tried different ways to study and invent calculus, and the invention finally led to a revolution in the world of math.

1. Introduction

C an a person chase up a turtle? This might sound like a very stupid question. However, this question had been disturbing mathematicians for many centuries.

In the history of Greek, Achilles was the hero who can run fastest in the world. In 5th century B.C., ancient Greek mathematician Zeno proposed an interesting paradox. He assumed that a turtle was 1 feet in front of a normal person, Achilles. They started to run on the same straight path at the same time. He also assumed that Achilles' velocity was 10 times of the turtle's, which means if the turtle ran at the velocity of v feet per second, Achilles could run at the velocity of 10v feet per second, meanwhile, the turtle could run 1/10 of the distance Achilles ran. In this case, when Achilles ran 1 feet and arrived at the starting point of the turtle, the turtle already has ran $l/10$ feet forward. When Achilles ran another $l/10$ feet, the turtle was $l/100$ feet in front of him. According to this assumption, whenever Achilles ran to the point where the turtle was at, the turtle would always be in front of him. The distance was getting shorter and shorter, but Achilles will never be able to chase up the turtle. At that time, although everybody knew that a normal person could chase up a turtle easily in reality, nobody could explain this paradox, until the theory of calculus had been developed.[1]

2. Calculus and Achilles Paradox

When I read about this paradox on the internet, I was taking AP Calculus at school. Such a challenging but interesting question really attracted me and made me keeping thinking, then a bold idea came up to my mind, I wondered if I could solve this paradox according to my knowledge of calculus. At this time, I started to figure out the question, expecting to obtain the explanation through Qualitative analysis and quantitative analysis. At the very beginning, I started with listing out all the knowledge we had learned in Calculus. Then, I used a set of terms to indicate the distance Achilles traveled each time to chase the turtle, l, $l/10$, $l/100$, $l/100^{n-1}$. I found out that it is a geometric sequence, a sequence made by multiplying by some fixed value each time. Then, this sequence reminded me the formula that helps me to calculate the sum of it, $\frac{a_1(1 - q^n)}{1 - q}$, where a_1 was the first number of the sequence,

Corresponding Author: *Zhantao Yang, Mailing Address: The Gilbert School, No.200 Williams Avenue, Winsted, CT 06098, USA. E-mail:2328462680@qq.com.*

q was the value that the sequence multiplied each time, and n is the total number of the term of the sequence. When I tried to use the formula to help me to solve the problem, I realized that according to what Zeno assumed, n was infinity, ∞. I substituted all corresponding numbers into the formula, the total distance needed to chase up the turtle was equal to $\frac{l(1-.1^\infty)}{1-.1}$. Since limit of a number between but not include -1 and 1 equaled to 0 when its power was ∞, we could simplify it into $l/.9$, which was further concluded that the total distance Achilles needed to chase up the turtle was $10l/9$ feet.

Finally, a mysterious and challenging question had been solved by only using high school and even lower leveled knowledge. Although I had successfully solved the problem, but a new confusion had been raised to me, why were mathematicians disturbed by such an easy question? And what eventually gave them the chance to solve the problem? When I read through the history of math and calculus, I found that the existence of calculus changed the way people knew about math. Calculus was the thing that helped them to solve the problem.

3. The Development of Calculus

3.1 The History of Calculus

Calculus, which included integration, differentiation and knowledge of limitation, did not just appear in a sudden. Scientists and mathematicians had been learning and discovering it for many centuries. At first, people knew about integration more, they used it more in science and normal life. In 3rd century B.C., ancient Greek scientist and mathematician Archimedes already discussed some methods and thoughts that are related to calculus in the books Measurement of a Circle and On the Sphere and the Cylinder, the introduction of finding the area under curves, sphere crowns and spiral, the volume of rotating hyperbola became the fundamental and starting parts of calculus. In 3rd century A.C., ancient Chinese mathematician Hui Liu also contributed to the development of calculus, his theory about cutting spheres and calculating the volume of objects was also a part of integration.

After that, although people were still trying to learn more about the knowledge of calculus, only little progress was made until 16th century. Luca Valerio, an Italian mathematician, restarted the discussion of the calculus in his book De quadratura parabolae due to the needs of it in scientific and mechanic problems. About the same time, Kepler, a German mathematician, proposed a way to calculate the area by thinking it as sums of lines, although

he was not able to tell a precise way to find the answer, this method gave the development of calculus a great inspiration. After Kepler, many other mathematicians and scientists further studied and improved the method. Cavalieri used his own method to calculate the sum of lines, he successfully figured out the area under the curve $y=x^n$, his method was further improved, and it was used to calculate the volume of some objects. Roberval used an easier way to calculate the area; he considered it as the sums of rectangle strips. This theory became an important leading of integration. Around then, Fermat made another big step forward. He not only made some assumptions and deductions about parabolas and hyperbolas, but also introduced the way to find maximum and minimum with the derivative that equaled to zero. His method is still widely used nowadays.[2] Other than them, Rene Descartes, Girard Desargues, Isaac Barrow, Evangelista Torricelli and many other mathematicians provided vital knowledge foundation for the establishment of calculus.

At the end of 17th century, calculus finally became a recognized math concept. Gottfried Wilhelm Leibniz and Isaac Newton were known as the major contributors to the development of calculus. Many people considered them as the inventors of calculus. However, most people believed that they developed it independently; they used different kinds of notations for calculus. The reason why they were called the inventor and major developer of calculus was related with the fact that they connected and summarized two concepts which had been never found before. One was tangent line, which was the major problem of differentiation. Another was quadrature, which was the major problem of integration.

3.2 Newton and Leibniz

It had been believed that Newton did not publish anything about calculus until 1693. According to Newton's claim, he did not publish anything because he was afraid of being ridiculed by people. As a result, his complete theory was first shown to people until 1704. In 1671, Newton wrote the book Methodus fluxionum et se-rierum infinitarum, the hand scripts were used among mathematicians for discussing and researching when Newton was still alive, but the English translation did not come out until 1736, after his death.[3] In this book, Newton talked about calculus in the problems of movement. He objected his previous opinion that a variable was a stationary set of infinitesimal elements; instead, he claimed that variables were produced by the continuous movement of points, lines and planes. He solved the problems of differentiation that gave the distance traveled and asked for instantaneous velocity. He also solved the problem of integration that gave the

velocity and needed to solve for distance traveled in time. Other than these, Newton also accomplished many other things in calculus, he wrote many reports and drew many important conclusions.

As for Leibniz, he was considered as the first person who published articles about calculus, the first calculus article was published in 1684. Although this article was ambiguous, it still talked about some major thoughts in calculus. In the article, calculus was defined firsttime, so it was considered as a huge step in the history of math. Leibniz also used some notations and rules in the article that people still use nowadays. His notations and rules were a lot easier to write and use than what Newton used. In the rest of his life, Leibniz kept studying calculus, he wrote down the thoughts and process that led him to invent calculus from 1714 to 1716, but the article was not published until 1846. Moreover, after his death, people found a lot of valuable notes about calculus, according to his notes, his first theory of differentiation was completed in 1675, which is earlier than Newton did.

Because Newton and Leibniz invented calculus almost at the same time, and both England and Germany wanted to make the invention theirs, many scholars argued about the inventor of calculus. Both of them started with infinitesimals, but Newton spent more time and researches on kinematic related problems, while Leibniz emphasized more about geometric problems. Also, there were lots of proofs in Leibniz' notes and articles that could tell he probably invented earlier than Newton. However, Leibniz was still doubted that he copied from Newton's work. Since 1695, many English scholars had called Newton the inventor of calculus, and German mathematicians followed with their opinion that it belonged to Leibniz. At that time, Newton was so well known, although he certainly contributed a lot in different fields, his knowledge and thoughts were still over admired by many people. Additionally, the Royal Society of England which had very high status claimed the invention belonged to Newton. Many people believed so; Leibniz was not as widely recognized until his death. In addition, England did not use Leibniz' easier ways of writing, instead, they used Newton's, which slowed down their process of science and math discovery. Nowadays, people admit both of their works, thanks to their invention; people are able to use calculus to solve many problems in not only science, but also daily life.[4]

Although Newton and Leibniz successfully established calculus system, but it was still not accurate and completed. In the next few centuries, people kept studying and improving calculus, which made it better.

4. Limit and Riemann Sum

4.1 Limit

But how did the theory of Newton, Leibniz and many other mathematicians helped to solve the problem of Achilles and the turtle? In their theory of calculus, there was a concept called limit. Generally, limit was when a variable that kept approaching but could never reach a value A. When it was close enough, we say it could never equal to the value A, but it is accurate enough to evaluate the value as A. This concept was not accurately written until Augustin Louis Cauchy and Karl Theodor Wilhelm Weierstraß. Although in Newton's and Leibniz' theory, the concept of limit was not accurately discussed, it was still enough to answer Zeno's paradox. In the paradox, it was true that Achilles needed infinity times to chase up the turtle, in the theory of limit, we could consider the times Achilles attempted as the variable that was approaching a value, the turtle, but he could never reach it. However, there was a limit, when Achilles was close enough, his distance traveled could be evaluated the same as the turtle's, so he could finally chase up. In other word, people used to commonly believe that if the person needed infinity attempts to chase up another, then the distance and time he needed to chase up was also infinity, which means he could never chase up. However, limit suggested infinity could lead to limited distance and time, limited distance and time could also lead to an infinity answer. Also, in our method, $(\frac{1}{2})^{\infty}$

equaled to 0 because the number was getting smaller and approached 0, so we say its limit is 0.[5]

After Achilles finally chased up the turtle, the knowledge of limit had helped people to solve more problems, and it also led to some other concepts and methods of calculation in calculus. There was a similar example of Achilles, when 1 is divided by 3, we can get 0.333..., it was an infinite decimal, we used it to multiply three, we got 0.999..., another infinite decimal that approached 1 but did not equal to 1. With the help of limit, the limit of 0.999... could be evaluated as 1 because it infinitively approached 1, in short, 0.999... is 1.

4.2 Riemann Sum

Another example of utilizing limit is Riemann sum. Although there was not specific proof of Riemann sum on text book, but I was still able to find out the reason using limit. Riemann sum was the method to calculate the area under curves by considering the area as many rectangles with the same width. Although there would be some dif-

ference, when the amount of rectangle increased, the difference decrease, the area of rectangle approached the original area, when infinite amount of rectangle summed up, we could say they had the same area. In this case, mathematicians introduced sigma, $\sum$ to further prove the Riemann sum. Generally, sigma meant the sum of a set of polynomials, $\sum_{i=m}^{b} c_b$=cm+cm+1+cm+2+...+cb which m was

the beginning of the polynomials, b was the end, and cb was the polynomial structure. In our problem, we said that the area was the sum of all the rectangles, and the number of rectangle was approaching infinity, so we could get , $\lim_{k \to \infty} \sum_{n=1}^{n=k} \Delta x f(x_n)$ which 1 was the first rectangle, k was the

last rectangle, Δx was the width between two closest points, which was also the same width of each rectangle, $f(xn)$ was the length of a rectangle, which would change in each of them. Then, $F(xn)$ was used to represent the y value at the point $x=xn$. According to Mean Value Theorem, which suggested $\frac{F(x_n) - F(x_{n-1})}{\Delta x}$ was the point of secant

line, which was $f(xn)$ when the interval was cut into infinite part. So, when I substituted, I got $\lim_{k \to \infty} \sum_{n=1}^{n=k} \Delta x \frac{F(x_n) - F(x_{n-1})}{\Delta x}$,

which could be further simplified, $\lim_{k \to \infty} \sum_{n=1}^{n=k} F(x_n) - F(x_{n-1})$.

Due to the definition of sigma, the formula equaled to $\lim_{k \to \infty} F(x_2) - F(x_1) + F(x_3) - F(x_2) + ... + F(x_k) - F(x_{k-1})$.

Then, I crossed out the plus and minus of the same polynomials, I got $\lim_{k \to \infty} F(x_k) - F(x_1)$, which equaled to $\int_{x_1}^{x_k} f(x)dx$ due to the definition of definite integration. So

far, I proved the definite integral, , $\int_{x_1}^{x_k} f(x)dx$ to indicate the actual area bounded by the the curve and the coordinate axis. Therefore, I had successfully proved the Riemann sum by myself with the knowledge we learnt in calculus class.

5. Conclusion

All in all, calculus had improved and changed our life slowly in thousands of years. Many difficult problems had been solved by using calculus, and by learning it, I had proven myself ready for more challenge and exciting problems. In my process of researching, I found there was a problem which is similar with the story of Achilles in the knowledge of philosophy. Although today we can use the knowledge of limitation and integration to explain that problem, but we still could not find a solution to the problem with the concepts in the field of philosophy. Calculus still has many interesting secrets to be found, and I will work on.

References

[1] Stancu D D. Rev Rournaine Math Pures Appl, 1968,13:1173-1194.

[2] Stancu D D. In: Proc Cont Constr Theory Funct, Budapest, 1972:443-455.

[3] Kelisky R P,Rivlin T J. Pacific J of Math 1967,21(3):511-521.

[4] S Karlin,H M Taylor. A First Course in Stochastic Processes.Second Edition,A Cadmic Press,1975:190-191.

[5] Ding Chuan-Song,Lee Pengyee. On Absotutely Henstok in Tegrable Function. Real Analysis Exchang,1987,12(2):39-45.